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A $\mathbb{P}_N \times \mathbb{P}_N$ Spectral Element Projection Method for the Unsteady Incompressible Navier-Stokes Equations

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Abstract. In this paper, we present a $\mathbb{P}_N \times \mathbb{P}_N$ spectral element method and a detailed comparison with existing methods for the unsteady incompressible Navier-Stokes equations. The main purpose of this work consists of: (i) detailed comparison and discussion of some recent developments of the temporal discretizations in the frame of spectral element approaches in space; (ii) construction of a stable $\mathbb{P}_N \times \mathbb{P}_N$ method together with a $\mathbb{P}_N \to \mathbb{P}_{N-2}$ post-filtering. The link of different methods will be clarified. The key feature of our method lies in that only one grid is needed for both velocity and pressure variables, which differs from most well-known solvers for the Navier-Stokes equations. Although not yet proven by rigorous theoretical analysis, the stability and accuracy of this one-grid spectral method are demonstrated by a series of numerical experiments.

AMS subject classifications: 65N35, 74S25, 76D07

Key words: $\mathbb{P}_N \times \mathbb{P}_N$, Navier-Stokes equations, spectral element methods.

1. Introduction

There have been numerous studies on the numerical approximation and its applications to the Navier-Stokes equations (NSE) since the past century. The Navier-Stokes equations have some features that make the construction of numerical method difficult. For example, the coupling of the velocity and pressure in the NSE brings a great difficulty in the numerical simulation of incompressible flows.

Generally, there are two principal ways to deal with this coupling in the time-dependent NSE. One way is to first keep the velocity and pressure coupled at the level of time discretization leading to a generalized Stokes problem, and then to apply the so-called Uzawa algorithm to the resulting algebraic system once the generalized Stokes problem is discretized in space. The Uzawa algorithm employs a block Gauss elimination in the discrete saddle-point problem to decouple the velocity from the pressure and yields two positive definite symmetric systems: one for the velocity and the other for the pressure (see e.g. [20] and the references therein). This decoupling procedure has been proven to be more attractive than a direct algorithm, however the classical Uzawa algorithm suffers

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from expensive computation of the pressure system as the pressure matrix involves the inverses of the Helmholtz systems. This disadvantage has been overcome by using an additional splitting technique, termed occasionally as "matrix factorization" [13], leading to a Poisson-like equation for the pressure. This approach has a common foundation with traditional projection-like splitting approaches which give a Poisson equation for the pressure except that, in the former case, the splitting is effected in the discrete form of the equations. Such an approach was analyzed and applied to several computations in the papers of Perot [22], Couzy *et al.* [7] and Fischer [10], but no rigorous error estimate in time is available. We will term here this kind of methods as Uzawa-based method. The disadvantage of the Uzawa-based method is that a discrete form of the Ladyshenskaya-Brezzi-Babuška condition (LBB condition, see e.g. [5]) must be satisfied for obtaining the unique discrete solution. The well-known $\mathbb{P}_N \times \mathbb{P}_{N-2}$ spectral element method (SEM), introduced in [21], addresses this problem through the use of compatible velocity and pressure spaces that are free of pressure spurious modes. There exist however some methods that make use other space pairs than $\mathbb{P}_N \times \mathbb{P}_{N-2}$, we refer to [3,6] for a detailed description of these methods.

Projection-type methods, introduced first by Chorin [8,9] and Temam [27] in the late 1960s, give another way to decouple the velocity and the pressure in the computation of unsteady incompressible flows. They are based on a particular time-discretization of the equations governing viscous incompressible flows, in which the viscosity and the incompressibility of the fluid are dealt within two separate steps. By doing that, the original problem is reformulated into two simpler problems. The projection algorithm can be classed into two families: classical fractional step methods and pressure-correction methods. The classical fractional step methods have only first order convergence rate due to the fact that they are basically an artificial compressibility technique [23,24]. Unlike Uzawa-based methods that preserve the original pressure boundary conditions, projection-type methods introduce implicitly new pressure boundary conditions. The inconsistent pressure boundary conditions usually give rise to numerical instability or/and reduce the accuracy of the scheme. Different choices of the pressure boundary condition have been discussed to improve the efficiency of this kind of methods (see [18] for instance). In a standard pressure-correction scheme the pressure accuracy can be at most of first-order in the L^2-norm, as shown by Strikwerda and Lee in [25]. Later, a modified pressure-correction scheme was introduced by Timmermans, Minnev and Van De Vosse [26], and analyzed by Guermond and Shen [15] in the semi-discrete form and by Huang and Xu [17] in the full discrete form. Optimal error estimates have been obtained [17] by assuming that the discrete velocity and pressure space pair satisfies the LBB condition. The role of the LBB conditions in the frame of the projection-type schemes has been an issue for a long time. We refer to [2,14] for recent detailed discussions in this sense. It should be emphasized that from the point of view of implementation, the LBB condition between the velocity and the pressure approximation space is not mandatory for the projection methods to work. Indeed, a principal interest in using the projection-type method is that we are free from the compatibility restriction on the choice of the discrete velocity and pressure space. Otherwise, the Uzawa-based method [7,10,19,22,29] could be the preference.

From the theoretical point of view, it is well-known that the LBB condition is a nec-