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Fourth-Order Splitting Methods for Time-Dependant Differential Equations

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Abstract. This study was suggested by previous work on the simulation of evolution equations with scale-dependent processes, e.g., wave-propagation or heat-transfer, that are modeled by wave equations or heat equations. Here, we study both parabolic and hyperbolic equations. We focus on ADI (alternating direction implicit) methods and LOD (locally one-dimensional) methods, which are standard splitting methods of lower order, e.g. second-order. Our aim is to develop higher-order ADI methods, which are performed by Richardson extrapolation, Crank-Nicolson methods and higher-order LOD methods, based on locally higher-order methods. We discuss the new theoretical results of the stability and consistency of the ADI methods. The main idea is to apply a higher-order time discretization and combine it with the ADI methods. We also discuss the discretization and splitting methods for first-order and second-order evolution equations. The stability analysis is given for the ADI method for first-order time derivatives and for the LOD (locally one-dimensional) methods for second-order time derivatives. The higher-order methods are unconditionally stable. Some numerical experiments verify our results.

AMS subject classifications: 35J60, 35J65, 65M99, 65N12, 65Z05, 74S10, 76R50, 80A20, 80M25

Key words: Partial differential equations, operator-splitting methods, evolution equations, ADI methods, LOD methods, stability analysis, higher-order methods.

1. Introduction

Using the classical operator-splitting methods, e.g., the Strang-Marchuk splitting method, we decouple the differential equation into more basic equations, so that each equation becomes simpler or contains only one operator, see [22, 23, 28]. These methods are often not sufficiently stable and also neglect the physical correlations between the operators, see [3]. The present work develops new efficient higher-order methods based on a stable variant of ADI or LOD methods, see [17, 20]. The decomposition ideas in exponential operators, see [3], which deal with the Sheng-Suzuki theorem, are based on the operator structure and conserve the physical characteristics. We contribute a further idea

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that uses stable and conserved lower-order methods, improves them with extrapolation methods and derive new higher-order results. The standard second-order method consequently applies Richardson extrapolation to obtain fourth-order and higher-order results. The physical decoupling in new operators and the deriving of new, strong directions are presented in the applications, see [3,16], and can be included in our theory. We consider a more abstract decomposition method, that is motivated by spatial and temporal directions. The theoretical results are obtained by application of the Neumann linear stability analysis, see [18]. For the LOD method we apply weak formulations for the stability results, see [20]. At the least we obtain higher-order methods in time and space and derive new stability results. We compared our results to methods discussed in [29], which dealt with global extrapolation of the parallel splitting method and obtained at least second and third order. Such methods are simple to implement but are not proposed to fourth order methods. We can also take into account the higher-order discretization in time and space. Our extrapolation methods are applied to lower-order ADI and LOD methods and can balance the underlying higher-order space and time discretizations. Therefore we can expand our splitting method to fourth-order methods in time and space. Moreover, our methods can be used for heat as well as wave equations. The theoretical results are verified by numerical experiments and examine the stability and consistency of the proposed methods.

The paper is organized as follows. A mathematical model and the underlying discretization methods for the heat and wave equation are introduced in Section 2. The splitting method for the evolution equations is given in Section 3. The stability analysis of the higher-order splitting method is given in Section 3.4. We discuss the numerical results in Section 4. Finally, we consider future works in the area of splitting and decomposition methods.

2. Mathematical model and discretization

The study presented below was suggested by various real-life problems whose governing equations are of evolution type. The first group presents the discussion of heat-transfer problems, see, e.g., [13], which are modeled by parabolic equations. The second group presents computational simulation of earthquakes, see, e.g., [4] and the examination of seismic waves, see [1,2]. Their underlying equations are hyperbolic differential equations.

2.1. Heat equation

Further, we have the heat equation, see [13], for which the mathematical equations are given by

$$\partial_t u = D_1(x,y)\,\partial_{xx} u + D_2(x,y)\,\partial_{yy} u + D_3(x,y)\,\partial_{zz} u, \text{ in } \Omega \times [0,T], \tag{2.1}$$

$$u(x,y,0) = u_0(x,y), \text{ on } \Omega, \tag{2.2}$$

The unknown function $u = u(x,t)$ is considered to be in $\Omega \times (0,T) \subset \mathbb{R}^d \times \mathbb{R}$ where the spatial dimension is given by d. The function $\mathbf{D}(x,y) = (D_1(x,y), D_2(x,y), D_3(x,y))^t \in$