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Development and Comparison of Numerical Fluxes for LWDG Methods

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Abstract. The discontinuous Galerkin (DG) or local discontinuous Galerkin (LDG) method is a spatial discretization procedure for convection-diffusion equations, which employs useful features from high resolution finite volume schemes, such as the exact or approximate Riemann solvers serving as numerical fluxes and limiters. The Lax-Wendroff time discretization procedure is an alternative method for time discretization to the popular total variation diminishing (TVD) Runge-Kutta time discretizations. In this paper, we develop fluxes for the method of DG with Lax-Wendroff time discretization procedure (LWDG) based on different numerical fluxes for finite volume or finite difference schemes, including the first-order monotone fluxes such as the Lax-Friedrichs flux, Godunov flux, the Engquist-Osher flux etc. and the second-order TVD fluxes. We systematically investigate the performance of the LWDG methods based on these different numerical fluxes for convection terms with the objective of obtaining better performance by choosing suitable numerical fluxes. The detailed numerical study is mainly performed for the one-dimensional system case, addressing the issues of CPU cost, accuracy, non-oscillatory property, and resolution of discontinuities. Numerical tests are also performed for two dimensional systems.

AMS subject classifications: 65M60, 65M99, 35L65

Key words: Discontinuous Galerkin method, Lax-Wendroff type time discretization, numerical flux, approximate Riemann solver, limiter, WENO scheme, high order accuracy.

1. Introduction

In this paper, we develop fluxes for the method of DG with Lax-Wendroff time discretization procedure (LWDG) based on different numerical fluxes for finite volume or finite difference schemes, including the first-order monotone fluxes such as the Lax-Friedrichs flux, Godunov flux, the Engquist-Osher flux etc. and the second-order TVD fluxes, and investigate the performance of the LWDG method based on different numerical fluxes for convection terms for solving nonlinear convection-diffusion scalar equations or systems:

$$\begin{cases} u_t + \nabla \cdot f(u) = \nabla \cdot fd(u, \nabla \cdot u), \\ u(x,0) = u_0(x), \end{cases} \tag{1.1}$$

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where f and fd are convection and diffusion terms, respectively, with the objective of obtaining better performance by choosing suitable numerical fluxes.

The discontinuous Galerkin (DG) method [3–7] for solving hyperbolic conservation laws and its extension to time-dependent convection-diffusion equations, the local DG (LDG) methods [1, 8, 9] are high order finite element methods employing the useful features from high resolution finite volume schemes, such as the exact or approximate Riemann solvers, and total variation bounded (TVB) limiters [26].

DG or LDG method is a spatial discretization procedure, namely, it is a procedure to approximate the spatial derivative terms in (1.1). The time derivative term can be discretized by explicit, nonlinearly stable high order Runge-Kutta time discretizations [25,27], and the scheme is termed as RKDG or RKLDG scheme, respectively. An alternative approach could be using a Lax-Wendroff type time discretization procedure, which is also called the Taylor type referring to a Taylor expansion in time or the Cauchy-Kowalewski type referring to the similar Cauchy-Kowalewski procedure in partial differential equations (PDEs) [28]. This approach is based on the idea of the classical Lax-Wendroff scheme [17], and it relies on converting all the time derivatives in a temporal Taylor expansion into spatial derivatives by repeatedly using the PDE and its differentiated versions. The spatial derivatives are then discretized by, e.g. the DG approximations. The methods are termed as LWDG methods for conservation laws [20]. Lax-Wendroff type time discretization procedure is also used by Dumbser and Munz [10, 11], in which they developed the ADER (Arbitrary high order schemes using DERivatives, see [29]) discontinuous Galerkin method using generalized Riemann solvers [29]. ADER methods also use the Lax-Wendroff procedure to convert time derivatives to spatial derivatives. The Lax-Wendroff type time discretization was also used in high order finite volume schemes [14, 29] and finite difference schemes [22].

As pointed out in [20], the LWDG is a one step, explicit, high order finite element method, the limiter is performed once every time step. As a result, LWDG is more compact than RKDG and the Lax-Wendroff time discretization procedure is more cost effective than the Runge-Kutta time discretizations for certain problems including two dimensional Euler systems of compressible gas dynamics when nonlinear limiters are applied.

An important component of the DG methods for solving conservation laws (1.1) is the numerical flux, based on exact or approximate Riemann solvers, which is borrowed from finite difference and finite volume methodologies. In most of the DG papers in the literature, the two-point, first order monotone Lax-Friedrichs (LF) numerical flux is used due to its simplicity. However, there exist many other numerical fluxes based on various approximate Riemann solvers in the literature, such as other two-point, first order monotone fluxes and essentially two-point TVD flux, which could also be used in the context of DG methods. The local LF (LLF) numerical flux, the Godunov flux [13], the Engquist-Osher (EO) flux [12, 18] for the scalar case and its extension to systems (often referred to as the Osher-Solomon flux [18]), the HLL flux [15] and a modification of the HLL flux, often referred to as the HLLC flux [31] are based on the approximate Riemann solver, these fluxes are two-point, first order monotone fluxes. One of the essentially two-point TVD fluxes is the flux limiter centered (FLIC) flux [30] with the following *essentially two-point* property: $\hat{f}(u^l, u, u, u^r) = f(u)$ for any u^l and u^r, which combines a low order monotone flux and a