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Spectral Method for Three-Dimensional Nonlinear Klein-Gordon Equation by Using Generalized Laguerre and Spherical Harmonic Functions

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Abstract. In this paper, a generalized Laguerre-spherical harmonic spectral method is proposed for the Cauchy problem of three-dimensional nonlinear Klein-Gordon equation. The goal is to make the numerical solutions to preserve the same conservation as that for the exact solution. The stability and convergence of the proposed scheme are proved. Numerical results demonstrate the efficiency of this approach. We also establish some basic results on the generalized Laguerre-spherical harmonic orthogonal approximation, which play an important role in spectral methods for various problems defined on the whole space and unbounded domains with spherical geometry.

AMS subject classifications: 65M70, 41A30, 81Q05

Key words: Generalized Laguerre-spherical harmonic spectral method, Cauchy problem of nonlinear Klein-Gordon equation.

1. Introduction

In this paper, we consider the nonlinear Klein-Gordon equation:

$$\partial_t^2 U(\mathbf{x},t) - \Delta U(\mathbf{x},t) + U(\mathbf{x},t) + U^3(\mathbf{x},t) = f(\mathbf{x},t), \tag{1.1}$$

which plays an important role in several fields, such as quantum mechanics, relativistic scalar field with power interaction, soliton theory and nonlinear meson theory of nuclear forces, see, e.g., [3, 15, 17, 19, 20]. The existence, uniqueness and regularity of its solution were studied, see, e.g., [17, 20]. On the other hand, many algorithms were proposed for its numerical simulation. In the early work, we usually employed finite difference method

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for periodical problems and non-periodical problems defined on bounded domains, see, e.g., [1, 16, 22] and the references therein. Some authors developed the Legendre spectral method for (1.1) defined on a finite interval, see [8]. In practice, it is also important and interesting to consider non-periodical problems in the whole multiple-dimensional space. We might use finite difference method for such problems. However, in this case, we have to confine our computation to certain bounded subdomain with artificial boundary condition. This treatment induces additional numerical errors. Thereby, it seems reasonable to solve (1.1) directly by using spectral method for the whole space. Whereas, so far, there is no result on the numerical solution of non-periodical Cauchy problem of multiple-dimensional nonlinear Klein-Gordon equation.

This work is concerned with numerical solution of the Cauchy problem of three dimensional nonlinear Klein-Gordon equation. Because of several reasons, we preferable to an alternative formulation in spherical coordinates. Let $\mathbf{x} = (x_1, x_2, x_3)^T$ and

$$x_1 = \rho \cos\lambda \cos\theta, \quad x_2 = \rho \sin\lambda \cos\theta, \quad x_3 = \rho \sin\theta.$$

Then (1.1) becomes

$$\begin{aligned}&\partial_t^2 U(\rho,\lambda,\theta,t) - \frac{1}{\rho^2}\partial_\rho\big(\rho^2\partial_\rho U(\rho,\lambda,\theta,t)\big) - \frac{1}{\rho^2\cos\theta}\partial_\theta\big(\cos\theta\,\partial_\theta U(\rho,\lambda,\theta,t)\big)\\&\quad - \frac{1}{\rho^2\cos^2\theta}\partial_\lambda^2 U(\rho,\lambda,\theta,t) + U(t,\rho,\lambda,\theta) + U^3(\rho,\lambda,\theta,t) = f(\rho,\lambda,\theta,t). \qquad (1.2)\end{aligned}$$

Since the longitude λ and the latitude θ vary on the finite intervals $[0, 2\pi]$ and $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$, respectively, there left only the variable ρ varying from 0 to ∞. This fact lowers the difficulties of calculation essentially. Moreover, we could adopt the spherical harmonic approximation in the longitude and latitude directions. Thereby, benefiting from the orthogonality of spherical harmonic functions, we simplify actual computation and numerical analysis.

The remaining problem is how to approximate (1.2) in the radial direction properly. It is natural to use certain orthogonal approximation on the half line. As we know, there have been already several kinds of Laguerre-type approximations. For instance, Funaro [5], Iranzo and Falques [14], Maday, Pernaud-Thomas and Vandeven [18], Guo and Shen [10], Guo and Xu [11], and Xu and Guo [23] developed the standard Laguerre approximation with its applications to partial differential equations defined on the half line and an infinite strip. Meanwhile, Guo, Shen and Xu [9], and Guo and Zhang [13] considered the generalized Laguerre approximation with its applications to two-dimensional exterior problems an so on.

In order to solve (1.2) efficiently, we need a specific orthogonal approximation, based on the main feature of (1.2).

• The nonlinear Klein-Gordon equation possesses certain conservation, which plays an important role in both theoretical analysis and numerical simulation. But the weight function $e^{-\rho}$ used in the standard Laguerre approximation destroys such property. Although we may reformulate (1.2) to a well-posed problems in a certain weighted space and then resolve the resulting problem by the standard Laguerre spectral method, it is much simpler