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Standard and Economical Cascadic Multigrid Methods for the Mortar Finite Element Methods

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Abstract. In this paper, standard and economical cascadic multigrid methods are considered for solving the algebraic systems resulting from the mortar finite element methods. Both cascadic multigrid methods do not need full elliptic regularity, so they can be used to tackle more general elliptic problems. Numerical experiments are reported to support our theory.

AMS subject classifications: 65F10, 65N30
Key words: Cascadic multigrid, mortar finite elements.

1. Introduction

Cascadic multigrid (CMG) [8, 9] is a type of multigrid methods which requires no coarse grid corrections at all that may be viewed as a "one way" multigrid. The main advantage of the cascadic multigrid method is its simplicity. Numerical experiments [9, 10] show that this method is very effective. Meanwhile, it has been proved [8,23] that the cascadic multigrid which uses the P1 conforming element for second-order elliptic problem in 3-D is accurate with optimal computational complexity for all one-step conventional iterative methods, like the weighted Jacobi, Gauss-Seidel and Richardson iteration as well as for the conjugate gradient method as a smoother. However, in 2-D case, the cascadic multigrid gives accurate solution with optimal computational complexity for the conjugate gradient method, but only nearly optimal complexity for other conventional iterative smoothers. In recent years, there have been several theoretical analysis and applications of these methods, cf. [24, 30] for nonconforming element methods and plate bending problems, [25] for parabolic problems, [21, 31] for nonlinear problems, [12] for Stokes problems, [14] for mortar element methods, [27] for the finite volume methods.

Recently, we proposed in [28] a new type of cascadic multigrid method. Compared with the standard cascadic multigrid method developed by Bornemann and Deuflhard

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[8,9], the new one requires less iterations on each level, especially on coarser grids. Many operations can be reduced in the new cascadic multigrid algorithms. So we call it an economical cascadic multigrid method (ECMG). It is proved that the new cascadic multigrid algorithm is still as optimal as the standard cascadic multigrid algorithm in both accuracy and computational complexity.

The mortar finite element method as a special domain decomposition methodology appears very attractive because it can handle situations where meshes on different subdomains need not align across interfaces, and the matching of discretizations on adjacent subdomains is only enforced weakly. In [7], Bernardi, Maday and Patera first introduced basic concepts of general mortar element methods, including the coupling of spectral elements with finite elements. Then it was extensively used and analyzed by many authors. In [5], Belgacem studied the mortar element method under a primal hybrid finite element formulation. Meanwhile, some extensions and convergence results in three dimensions were considered in [4, 6]. Recently, many works have been done in constructing efficient iterative solvers for the discrete system resulting from the mortar element method. The first approaches were based on iterative substructuring methods, see [1–3, 18]. Multigrid methods for the mortar element methods have also been considered. Gopalakrishnan and Pasciak [20] presented a variable V-cycle preconditioner, while Braess, Dahmen and Wieners [13] and Wohlmuth [29] established a W-cycle multigrid based on a hybrid formulation which gives rise to a saddle point problem.

The objective of this paper is to design efficient cascadic multigrid (CMG) solvers for the mortar finite element methods. Note that Braess, Deuflhard and Lipnikov [14] constructed a subspace cascadic multigrid method for the mortar element method based on a saddle point formulation. Moreover, the authors only considered second-order elliptic problems with full regularity. In this paper, we will treat the mortar element method under the framework of nonconforming methods and assume that the Lagrange multiplier has been eliminated. We will construct the standard and economical cascadic multigrid methods for solving the algebraic system resulting from such kind of mortar finite element method and then give their convergence analysis for more general second elliptic problems without full regularity such as L-shape domain problems.

This paper is organized as follows: Section 2 introduces the mortar element method developed by Bernardi, Maday and Patera in [7]. In Section 3, we give the standard cascadic multigrid method and its convergence analysis. In Section 4, we introduce the economical cascadic multigrid method. Finally, numerical results that confirm our theory will be given in the Section 5.

2. The mortar element method

The mortar finite element method allows the coupling of different discretizations across subdomain boundaries. The idea of the mortar finite element method is to weakly impose the transmission conditions across the interface of difference subdomains by means of Lagrange multiplier. The key argument is to construct a suitable discrete Lagrange multiplier space in order to ensure the stability of the discrete problem. It is interesting to design some