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A Power Penalty Approach to Numerical Solutions of Two-Asset American Options

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Abstract. This paper aims to develop a power penalty method for a linear parabolic variational inequality (VI) in two spatial dimensions governing the two-asset American option valuation. This method yields a two-dimensional nonlinear parabolic PDE containing a power penalty term with penalty constant $\lambda > 1$ and a power parameter $k > 0$. We show that the nonlinear PDE is uniquely solvable and the solution of the PDE converges to that of the VI at the rate of order $\mathcal{O}(\lambda^{-k/2})$. A fitted finite volume method is designed to solve the nonlinear PDE, and some numerical experiments are performed to illustrate the usefulness of this method.

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1. Introduction

An option is a contract tradable in a financial market which gives to its owner the right to buy (*call option*) or to sell (*put option*) a fixed quantity of a specified asset or stock at a fixed price (*exercise* or *strike price*) on (European option) or before (American option) a given date (*expiry date*). The market prices of the rights to buy and to sell are called *call prices* and *put prices*, respectively. Clearly, the price of an option is dependent on the market price(s) of its underlying stock(s). How to valuate an option has long been a hot topic for financial engineers, economists and mathematicians. In the case of a European type option on a single asset, it was shown by Black and Scholes (cf. [4]) that the price satisfies a second-order partial differential equation with respect to the time horizon t and the underlying asset price x, known as the Black-Scholes equation. The value of an

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American option is determined by a linear complementarity problem involving the Black-Scholes operator [19]. Since this complementarity problem is, in general, not analytically solvable, numerical approximation to the solution is normally sought in practice. Various numerical techniques have been proposed for the numerical solution of the single-asset American option pricing problem. Among them, lattice method [6], explicit method [13], projected successive over relaxed method (PSOR) [14], linear programming method [7], Monte-Carlo method [5], and penalty method [2, 8, 9, 19–21] are the most popular ones in both practice and research.

A linear penalty approach to the linear complementarity problem was proposed and analyzed in [3] which has been used in [9, 21]. Compared with other methods mentioned above, the penalty method possesses several advantages. First, a desirable accuracy in the approximate solution can be achieved by a judicious choice of the penalty parameter. Second, the resulting penalized PDE is of a simple form that is easy to discretize in any dimensions on both structured and unstructured meshes. Finally, the penalty method can easily be extended to other option models such as those of American options with stochastic volatilities and/or transaction costs.

In the application of the penalty approach to American option pricing, a penalty term is added to the Black-Scholes equation. In [9], an l_1 penalty method is used, resulting in a convergence rate of order $\mathcal{O}(\lambda^{-1/2})$, where $\lambda > 1$ denotes the penalty parameter. A power penalty method is proposed and analyzed in [19, 20], of which the convergence rate is shown to be of order $\mathcal{O}(\lambda^{-k/2})$ for any power parameter $k > 0$. This contains the l_1 case as the special one when $k = 1$ and provides an exponential convergence rate when $k > 1$.

For a single asset American option, it has been shown in [1, 9, 19, 20] that the solution to the penalized equation converges to that of the original problem. To our best knowledge, there are no advances in the use of penalty methods for two dimensional problems in the open literature except for the quadratic and l_1 penalty methods for solving the American option pricing problem with stochastic volatility (cf. [21]) period. On the other hand, it has been shown in [20] that the penalty methods are superior to other methods such as PSOR mentioned above.

The main purpose of this paper is to develop a power penalty method for the linear complementarity problem arising from the two-dimensional American option valuation, which comes from many models, such as stochastic volatility model, interest rate model, basket options model, and so on (cf. [17]). Without loss of generality, in this paper we put our focus on the two-asset basket option model. We will approximate the linear complementarity problem by a nonlinear parabolic PDEs in two spatial dimensions with an l_k penalty term. We will then show that the solution to the nonlinear PDE converges to that of the original complementarity problem at the rate of order $\mathcal{O}(\lambda^{-k/2})$. To solve the penalized nonlinear equation, the fitted finite volume method is proposed, based on the results in [12, 18–20]. Numerical results will be presented to verify our theoretical findings.

The paper is organized as follows. In the next section, the pricing of two-asset American options will be formulated as a linear complementarity problem. This complementarity problem will, in Section 3, be reformulated as a variational inequality problem in a functional setting, and its unique solvability will be established as well. The power penalty