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An Efficient and Accurate Spectral Method for Acoustic Scattering in Elliptic Domains

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Abstract. An efficient and accurate method for solving the two-dimensional Helmholtz equation in domains exterior to elongated obstacles is developed in this paper. The method is based on the so called transformed field expansion (TFE) coupled with a spectral-Galerkin solver for elliptical domain using Mathieu functions. Numerical results are presented to show the accuracy and stability of the proposed method.

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1. Introduction

Many scientific and engineering applications require fast and accurate numerical approximation of acoustic and electromagnetic scattering that returns from irregular obstacles. Although the governing equation is linear, its numerical approximation presents a number of notorious difficulties: (i) the problem is set in an unbounded exterior domain, making it difficult to obtain accurate approximations when an artificial boundary is introduced; (ii) the problem is indefinite, making it difficult to design efficient iterative methods; and (iii) the solution is highly oscillatory when the incoming wave has high frequencies, making it inefficient to use low-order finite difference or finite element methods. A wide variety of numerical methods have been proposed to deal with these difficulties (cf. the review papers [14, 19] and the references therein). A particularly compelling class of methods are based on the boundary perturbation technique originated from the work of Rayleigh [13] and Rice [15], and we refer to [2–4] for some recent developments in this direction. More recently, a robust and accurate numerical method based on the transformed field expansion and a fast spectral-Galerkin solver is proposed for two- and

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three-dimensional acoustic scattering [5, 9, 11, 12]. The method has proven to be very efficient for obstacles that can be considered as a perturbation of a disk in 2-D or a sphere in 3-D. While in principle the algorithms in [5, 9] can be applied to elongated scatters (e.g., submarines and airfoils), which are found in many important applications, it may not be computationally efficient to do so due to the fact that large artificial boundaries are needed to enclosed the elongated obstacles. In such cases, it is more appropriate to use elliptic and ellipsoidal artificial surfaces to truncate the unbounded computational domains.

The purpose of this paper is to develop an efficient and accurate numerical method for the acoustic scattering from an elongated obstacle. The basic idea is to consider an elongated obstacle as a perturbation of ellipse in 2-D and of ellipsoid in 3-D, use a larger ellipse or ellipsoid to enclose the obstacle and reduce the problem to a bounded domain through the Dirichlet-to-Neumann mapping, and then develop an efficient and accurate spectral method for the reduced equation in the separated elliptic domain.

While spectral methods for partial differential equations in circular and spherical domains have been well developed, their applications to elliptical domains have received very little attention. The main reason is that the separation of variables in elliptical domains leads to Mathieu functions in 2-D and spheroidal wave functions in 3-D. However, the use of elliptic coordinates and Mathieu functions introduces significant difficulties in both analysis and implementation. Although the Mathieu functions have been the subjects of many studies (cf. [1, 7, 8]), most of which are concerned with their classical properties such as identities, recurrence and asymptotic relations. As far as we know, there were essentially no results on their approximation properties in Sobolev spaces which are required for numerical analysis of spectral methods using these special functions. In a very recent paper [18], two of the authors made a systematical study for the approximation properties of Mathieu functions and applied them to study the elliptic equations in a bounded separable elliptic domain. The analytical and numerical results presented in [18] indicate that Mathieu functions have nice approximation properties similar to those enjoyed by classical trigonometric polynomials and are suitable for numerical approximation of PDEs in elliptic domains. Hence, we shall use Mathieu functions as basis functions for the spectral-Galerkin solver in our scheme.

The rest of the paper is organized as follow. In Section 2, we describe the governing equation for acoustic scattering in exterior domains with elliptical coordinates and use the Dirichlet-to-Neumann mapping to reduce the problem to a bounded domain. We derive in Section 3 the transformed field expansion in the elliptical coordinates. Then, we construct a spectral-Galerkin method for solving the reduced Helmtoltz problem in a regular elliptical domain. We present some illustrative numerical results in Section 5 and conclude with some remarks in the last section.

2. Governing equation and Dirichlet-to-Neumann mapping

2.1. Governing equation

Consider a two-dimensional time-harmonic acoustic plane wave