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High-Order Leap-Frog Based Discontinuous Galerkin Method for the Time-Domain Maxwell Equations on Non-Conforming Simplicial Meshes

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Abstract. A high-order leap-frog based non-dissipative discontinuous Galerkin time-domain method for solving Maxwell's equations is introduced and analyzed. The proposed method combines a centered approximation for the evaluation of fluxes at the interface between neighboring elements, with a Nth-order leap-frog time scheme. Moreover, the interpolation degree is defined at the element level and the mesh is refined locally in a non-conforming way resulting in arbitrary level hanging nodes. The method is proved to be stable under some CFL-like condition on the time step. The convergence of the semi-discrete approximation to Maxwell's equations is established rigorously and bounds on the global divergence error are provided. Numerical experiments with high-order elements show the potential of the method.

AMS subject classifications: 65M12, 65M50, 65M60, 74S10, 78A40

Key words: Maxwell's equations, discontinuous Galerkin method, leap-frog time scheme, stability, convergence, non-conforming meshes, high-order accuracy.

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1. Introduction

The accurate modeling of systems involving electromagnetic waves, in particular through the resolution of the time-domain Maxwell equations on space grids, remains of strategic interest for many technologies. The still prominent Finite Difference Time-Domain (FDTD) method proposed by Yee [20] lacks two important features to be fully applied in industrial contexts. First, it has huge restriction to structured or block-structured grids. Second, the efficiency of FDTD methods is limited when fully curvilinear coordinates are used. Many different types of methods have been proposed in order to handle complex geometries and heterogeneous media by dealing with unstructured tetrahedral meshes, including, for example, mass lumped Finite Element Time-Domain (FETD) methods [12,14], mimetic methods [11], or Finite Volume Time-Domain (FVTD) methods [17], which all fail in being at the same time efficient, easily extendible to high orders of accuracy, stable, and energy-conserving.

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Recently, discontinuous Galerkin methods have attracted much research to solve electromagnetic wave propagation problems. Being higher order versions of traditional finite volume methods [13], Discontinuous Galerkin Time-Domain (DGTD) methods based on discontinuous finite element spaces, easily handle elements of various types and shapes, irregular non-conforming meshes [8, 9], and even locally varying polynomial degree [8]. They hence offer great flexibility in the mesh design, but also lead to (block-)diagonal mass matrices and therefore yield fully explicit, inherently parallel methods when coupled with explicit time stepping [1]. Moreover, continuity is weakly enforced across mesh interfaces by adding suitable bilinear forms (so-called numerical fluxes) to the standard variational formulations. Whereas high-order DGTD methods have been developed on conforming meshes [4,5,10], the design of non-conforming discontinuous Galerkin time-domain methods is still in its infancy. In practice, the non-conformity can result from a local refinement of the mesh (*i.e.*, h-refinement), of the interpolation degree (*i.e.*, p-enrichment) or of both of them (*i.e.*, hp-refinement).

This work is concerned with the study of high-order leap-frog schemes that are extensions of the second-order leap-frog scheme adopted in the DGTD methods that are studied in [8, 9]. The motivation behind this study is to improve the overall accuracy for the same mesh resolution and/or to improve convergence when the mesh resolution is increased. Not surprisingly, the arbitrary high-order DGTD methods discussed in this work are consistently more accurate than the DGTD methods based on the second-order leap-frog scheme. The high-order leap-frog schemes require more computational operations to update a cell. Fortunately, this can be alleviated by the ability to use discretization meshes with fewer points per wavelength for the same level of accuracy.

This paper is structured as follows. In Section 2, we introduce the high-order non-conforming DGTD method for solving the system of Maxwell's equations. Our two main results, the stability and the hp-convergence of the proposed method, are stated and proved in Section 3. In this section we also establish bounds on the behavior of the divergence error. In Section 4 we verify our theoretical results through numerical experiments. Finally, some concluding remarks are presented in Section 5.

2. An arbitrary high-order non-conforming DGTD method

We consider the Maxwell equations in three space dimensions for heterogeneous anisotropic linear media with no source. The electric permittivity tensor $\bar{\bar{\epsilon}}(x)$ and the magnetic permeability tensor $\bar{\bar{\mu}}(x)$ are varying in space, time-invariant and both symmetric positive definite. The electric field $\vec{\mathbf{E}}$ and the magnetic field $\vec{\mathbf{H}}$ verify:

$$\bar{\bar{\epsilon}}\partial_t\vec{\mathbf{E}} = \mathrm{curl}\,\vec{\mathbf{H}}, \quad \bar{\bar{\mu}}\partial_t\vec{\mathbf{H}} = -\mathrm{curl}\,\vec{\mathbf{E}}, \tag{2.1}$$

$$\mathrm{div}\,(\bar{\bar{\epsilon}}\vec{\mathbf{E}}) = 0, \qquad \mathrm{div}\,(\bar{\bar{\mu}}\vec{\mathbf{H}}) = 0, \tag{2.2}$$

where the symbol ∂_t denotes a time derivative. These equations are set and solved on a bounded polyhedral domain Ω of $\mathbb{R}^3$. For the sake of simplicity, a metallic boundary condition is set everywhere on the domain boundary $\partial\Omega$, *i.e.*, $\vec{n}\times\vec{\mathbf{E}} = 0$ (where $\vec{n}$ denotes the unitary outwards normal).