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Error Estimate of the Fourier Collocation Method for the Benjamin-Ono Equation

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Abstract. In this paper, the Fourier collocation method for solving the generalized Benjamin-Ono equation with periodic boundary conditions is analyzed. Stability of the semi-discrete scheme is proved and error estimate in $H^{1/2}$-norm is obtained.

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Key words: Fourier collocation, Benjamin-Ono equation, error estimate.

1. Introduction

In this paper, we analyze the Fourier collocation (FC) approximation to the generalized Benjamin–Ono (BO) equation with periodic boundary conditions:

$$\begin{cases} \partial_t U(x,t) + \partial_x F(U)(x,t) + \mathscr{H}\partial_x^2 U(x,t) = 0, & x \in \mathbb{R}, 0 < t \le T, \\ U(x+2\pi,t) = U(x,t), & x \in \mathbb{R}, 0 < t \le T, \\ U(x,0) = U_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where U_0 is 2π-periodic in space, $F(z) \in C^1(\mathbb{R})$, and $\mathscr{H}$ is the periodic Hilbert transform [1]

$$\mathscr{H}u(x) = -\frac{1}{2\pi}\mathrm{PV}\int_{-\pi}^{\pi} \cot\Big(\frac{\pi(x-y)}{2\pi}\Big)u(y)\mathrm{d}y.$$

The problem (1.1) arises in the propagation of internal waves in a stratified fluid of great depth. The special case $F(U) = U^2$ is the BO equation. Fourier methods for the BO equation have been studied by many authors [4, 9–12]. In recent work [11], it is proved that error of the Fourier Galerkin (FG) method for the BO equation is of the order $\mathscr{O}(N^{1-r})$ in L^2-norm for the analytic solution in H^r. An optimal error bound $\mathscr{O}(N^{1/2-r})$ of the method in $H^{1/2}$-norm is obtained in [4].

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As suggested in [6, 9, 10], the FC methods for the BO equation are efficient, but no error analysis has been provided. The aim of this work is to give rigorous proof of error estimate of the FC method for (1.1). In particular, it will be shown that the error is of the order $\mathcal{O}(N^{3/2-r})$ in $H^{1/2}$-norm.

In Section 2, the FC method for (1.1) is presented. In Section 3, some lemmas needed in error analysis are given. In Section 4, the stability and convergence of the semi-discrete FC method are analyzed. This paper does not give the analysis for the fully discrete scheme, but the accuracy of the fully discrete scheme will be demonstrated by using an example for the BO equation in Section 5.

2. The Fourier collocation method

Let $I=(-\pi,\pi)$. The inner product of $L^2(I)$ is denoted by $(\cdot,\cdot)$. For a positive integer N, the approximation space V_N of the real trigonometric polynomials of degree N is defined by

$$V_N=\left\{u(x)=\sum_{l=-N}^{N}{}'' a_l e^{ilx}\ :\ \overline{a_l}=a_{-l},\ |l|\le N;\ \ a_N=a_{-N}\right\},$$

where the notation $\sum''$ denotes halving the terms a_{-N} and a_N in the series. Let $h=2\pi/2N$, $x_j=jh-\pi$ $(j=0,\cdots,2N-1)$ be the collocation points so that the base 2 Fast Fourier Transform (FFT) can be directly adopted. Let $I_N: C(\bar{I})\to V_N$ be the Fourier interpolation operator defined by

$$I_N u(x_j)=u(x_j),\quad j=0,\cdots,2N-1.$$

We define the discrete product and norm as follows:

$$(u,v)_N=h\sum_{j=0}^{2N-1}u(x_j)\overline{v(x_j)},\qquad \|u\|_N=(u,u)_N^{1/2}.$$

Let $P_N: L^2(I)\to V_N$ be the L^2-orthogonal projection operator, i.e.,

$$(P_N u-u,v)=0,\qquad v\in V_N.$$

The semi-discrete FC method for (1.1) is to find $u_c(t)\in V_N$ such that for $0\le j\le 2N-1$,

$$\begin{cases}(\partial_t u_c+\partial_x I_N F(u_c)+\mathscr{H}\partial_x^2 u_c)(x_j,t)=0, & 0<t\le T,\\ u_c(x_j,0)=P_N U_0(x_j).\end{cases}\tag{2.1}$$

For the time advance, we use the second-order leapfrog-Crank-Nicolson scheme. Let τ be the step size in time and $t_k=k\tau$ $(k=0,1,\cdots,n_{_T};\ T=n_{_T}\tau)$. Denote $u^k(x):=u(x,t_k)$ by u^k and

$$u_{\hat{t}}^k=\frac{1}{2\tau}(u^{k+1}-u^{k-1}),\qquad \hat{u}^k=\frac{1}{2}(u^{k+1}+u^{k-1}).$$