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Convex Variational Formulation with Smooth Coupling for Multicomponent Signal Decomposition and Recovery

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Abstract. A convex variational formulation is proposed to solve multicomponent signal processing problems in Hilbert spaces. The cost function consists of a separable term, in which each component is modeled through its own potential, and of a coupling term, in which constraints on linear transformations of the components are penalized with smooth functionals. An algorithm with guaranteed weak convergence to a solution to the problem is provided. Various multicomponent signal decomposition and recovery applications are discussed.

AMS subject classifications: 94A12, 90C25

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1. Problem statement

The processing of multicomponent signals has become increasingly important due, on the one hand, to the development of new imaging modalities and sensing devices, and, on the other hand, to the introduction of sophisticated mathematical models to represent complex signals. It is for instance required in applications dealing with the recovery of multichannel signals [8, 33, 34, 40], which arise in particular in color imaging and in the multi- and hyperspectral imaging techniques used in astronomy and in satellite imaging. Another important instance of multicomponent processing is found in signal decomposition problems, e.g., [2,5–7,15,43,44]. In such problems, the ideal signal is viewed as a mixture of elementary components that need to be identified individually.

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Mathematically, a multicomponent signal can be viewed as an m-tuple $(x_i)_{1\le i\le m}$, where each component x_i lies in a real Hilbert space $\mathcal{H}_i$. A generic convex variational formulation for solving multicomponent signal recovery or decomposition problems is

$$\underset{x_1\in\mathcal{H}_1,\cdots,x_m\in\mathcal{H}_m}{\text{minimize}} \quad \Phi(x_1,\cdots,x_m), \tag{1.1}$$

where $\Phi\colon \mathcal{H}_1\oplus\cdots\oplus\mathcal{H}_m \to \left]-\infty,+\infty\right]$ is a convex cost function. At this level of generality, however, no algorithm exists to solve (1.1) reliably in the sense that it produces m sequences $(x_{1,n})_{n\in\mathbb{N}}$, $\cdots$, $(x_{m,n})_{n\in\mathbb{N}}$ converging (weakly or strongly) to points x_1, $\cdots$, x_m, respectively, such that $(x_i)_{1\le i\le m}$ minimizes Φ. Let us recall that, even in the elementary case when $m=2$ and $\mathcal{H}_1=\mathcal{H}_2=\mathbb{R}$, the basic Gauss-Seidel alternating minimization algorithm does not possess this property [28]. In this paper, we consider the following, more structured version of (1.1).

Problem 1.1. *Let $m\ge 2$ and $p\ge 1$ be integers, let $(\mathcal{H}_i)_{1\le i\le m}$ and $(\mathcal{G}_k)_{1\le k\le p}$ be real Hilbert spaces, and let $(\tau_k)_{1\le k\le p}$ be in $\left]0,+\infty\right[$. For every $i\in\{1,\cdots,m\}$, let $f_i\colon\mathcal{H}_i\to\left]-\infty,+\infty\right]$ be a proper lower semicontinuous convex function and, for every $k\in\{1,\cdots,p\}$, let $\varphi_k\colon\mathcal{G}_k\to\mathbb{R}$ be convex and differentiable with a τ_k-Lipschitz continuous gradient, and let $L_{ki}\colon\mathcal{H}_i\to\mathcal{G}_k$ be linear and bounded. It is assumed that* $\min_{1\le k\le p}\sum_{i=1}^m\|L_{ki}\|^2>0$. *The problem is to*

$$\underset{x_1\in\mathcal{H}_1,\cdots,x_m\in\mathcal{H}_m}{\text{minimize}} \quad \sum_{i=1}^m f_i(x_i)+\sum_{k=1}^p\varphi_k\left(\sum_{i=1}^m L_{ki}x_i\right), \tag{1.2}$$

under the assumption that solutions exist.

Let us note that (1.2) is a particular case of (1.1), in which Φ is decomposed in two terms, namely

$$\Phi(x_1,\cdots,x_m)=\underbrace{\sum_{i=1}^m f_i(x_i)}_{\text{separable term}}+\underbrace{\sum_{k=1}^p\varphi_k\left(\sum_{i=1}^m L_{ki}x_i\right)}_{\text{coupling term}}. \tag{1.3}$$

Each function f_i in the separable term promotes an intrinsic property of the ith component x_i of the signal. On the other hand, the coupling term models p interactions between the m components $(x_i)_{1\le i\le m}$. An elementary interaction is associated with a potential φ_k acting on a linear transformation $\sum_{i=1}^m L_{ki}x_i$ of the components. The coupling is smooth in the sense that the function φ_k is differentiable with a Lipschitz gradient. As will be seen in subsequent sections, Problem 1.1 not only captures existing formulations for which reliable solution methods are not available, but it also allows us to investigate a wide range of new problems. In addition, it can be solved reliably by the following proximal algorithm recently developed in [4] (the definition of the proximity operator $\operatorname{prox}_{f_i}$ of a convex function $f_i\colon\mathcal{H}_i\to\left]-\infty,+\infty\right]$ is given in Section 2.2).

Algorithm 1.1. *Set*

$$\beta_1=\frac{1}{p\max\limits_{1\le k\le p}\tau_k\sum_{i=1}^m\|L_{ki}\|^2}, \tag{1.4}$$