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A-Posteriori Error Estimation for the Legendre Spectral Galerkin Method in One-Dimension

Lijun Yi[1] and Benqi Guo[1,2,*]

[1] *Department of Mathematics, Shanghai Normal University, Shanghai, 200234, China.*
[2] *Department of Mathematics, University of Manitoba, Winnipeg, MB R3T 2N2, Canada.*

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Abstract. In this paper, a-posteriori error estimators are proposed for the Legendre spectral Galerkin method for two-point boundary value problems. The key idea is to postprocess the Galerkin approximation, and the analysis shows that the postprocess improves the order of convergence. Consequently, we obtain asymptotically exact a-posteriori error estimators based on the postprocessing results. Numerical examples are included to illustrate the theoretical analysis.

AMS subject classifications: 65N35, 65N30

Key words: Legendre spectral Galerkin method, two-point boundary value problem, superconvergence, a-posteriori error estimation.

1. Introduction

A-posteriori error estimation has now become an accepted and even expected tool in modern scientific computing. Many a-posteriori error estimators have been developed for low order finite element methods (FEM) (see, e.g., [1, 4, 27], and references therein), which are mainly based on the residual method [2, 3, 5, 8] or on the recovery method [35, 36].

In contrast to the low order FEM (h-FEM), a-posteriori error estimation for high order methods such as the spectral methods, the p-version FEM and the hp-version FEM is much less developed and lacks of substantial progress in the past two decades. There are only few papers on this topic in the current literature, see, e.g., [9, 12–16, 22–24].

In the present paper, we develop a-posteriori error estimation for the Legendre spectral Galerkin method [10, 13, 25, 29] for a certain class of two-point boundary value problems. We first construct a semi-H^1 projection which plays an important role in the analysis of

*Corresponding author. *Email addresses:* `ylj5152@hotmail.com` (L. J. Yi), `guo@cc.umanitoba.ca` (B. Q. Guo)

high order methods in one space dimension, and investigate it's approximation property. Following the classical superconvergence analysis of the h-FEM [4, 11, 21, 28, 32, 34], some superconvergence results are obtained. Then, we propose a postprocessing technique to enhance the Legendre spectral Galerkin method. It is proved that the postprocess improves the order of convergence of the Galerkin approximation. Actually, the postprocessing technique is essentially a correction for the Galerkin approximation, and the correction scheme (4.5) in section 4 shows that the correction quantities to the numerical solution can be expressed by a sum of some higher order polynomials, and the overcost of the postprocessing procedure is nearly negligible. Finally, it is possible to define recovery-based a-posteriori error estimators that are asymptotically exact by using the postprocessing results.

This paper is organized as follows: in the next section we present the model problem and construct the Legendre spectral Galerkin approximation scheme. In Section 3, we investigate the approximation properties of the semi-H^1 projection. In Section 4, we propose a postprocessing technique for the Galerkin approximation and the asymptotically exact a-posteriori error estimators are analyzed. The analytical results are illustrated by numerical examples in Section 5. We summarize the work and also discuss some possible future works in the last section.

Let $I \subset \mathbb{R}$ be an open and bounded interval. In this paper, we adopt the standard notation $W^{m,q}(I)$ for Sobolev spaces on I with the norm $\|\cdot\|_{m,q}$ and the seminorm $|\cdot|_{m,q}$. In addition, We denote $W^{m,2}(I)$, $W_0^{m,2}(I)$ by $H^m(I)$, $H_0^m(I)$, respectively. Hereafter, we denote by C a generic positive constant independent of any function and N, the order of the Galerkin approximation.

2. Legendre spectral Galerkin method

We consider the following two-point boundary value problem

$$\begin{cases} -u''(x)+b(x)u(x)=f(x), & \text{in } I=(-1,1), \\ u(-1)=u(1)=0, \end{cases} \tag{2.1}$$

with $b(x)\geq 0$, and we assume that b and f are sufficiently smooth for our analysis.

The weak form of (2.1) is to find $u\in H_0^1(I)$ such that

$$a(u,v)=(f,v), \quad \forall v\in H_0^1(I),$$

where

$$a(u,v)=\int_I (u'v'+buv)dx, \quad (f,v)=\int_I fv dx.$$

Let

$$L_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}[(x^2-1)^n], \quad n\geq 0$$