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A Modified Polynomial Preserving Recovery and Its Applications to A Posteriori Error Estimates

Haijun Wu*

Department of Mathematics, Nanjing University, Jiangsu, 210093, China.

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Abstract. A modified polynomial preserving gradient recovery technique is proposed. Unlike the polynomial preserving gradient recovery technique, the gradient recovered with the modified polynomial preserving recovery (MPPR) is constructed element-wise, and it is discontinuous across the interior edges. One advantage of the MPPR technique is that the implementation is easier when adaptive meshes are involved. Superconvergence results of the gradient recovered with MPPR are proved for finite element methods for elliptic boundary problems and eigenvalue problems under adaptive meshes. The MPPR is applied to adaptive finite element methods to construct asymptotic exact a posteriori error estimates. Numerical tests are provided to examine the theoretical results and the effectiveness of the adaptive finite element algorithms.

AMS subject classifications: 65N30, 65N15, 45K20

Key words: Adaptive finite element method, superconvergence, gradient recovery, modified PPR.

1. Introduction

Gradient recovery has been widely used for a posteriori error estimates (see, e.g., [1, 4, 9, 12, 20, 21, 25, 26, 32, 34–37]). Comparing with the a posteriori error estimates of residual type (see, e.g., [1,2,7,10,16,17,22,24]), the a posteriori error estimates based on gradient recovery have the advantages of problem-independence and asymptotic exactness. Although the effectiveness of using the a posteriori error estimates based on gradient recovery in adaptive finite element methods have been demonstrated by many practical applications, most theoretical results assume uniform meshes and sufficiently smooth solutions (see, e.g., [5,8,14,15,18,27,28,30,31,33]). Recently, Wu and Zhang [25] consider adaptive finite element methods for elliptic problems with domain corner singularities and prove a superconvergence result for recovered gradient by the polynomial preserving recovery (PPR) as well as the asymptotic exactness of the a posteriori error estimate based on PPR under two mesh conditions. One condition is similar to but weaker than the *Condition* (α,σ) used for uniform meshes (cf. [18,27,30,31]). Another one is a mesh density

*Corresponding author. *Email address:* `hjw@nju.edu.cn` (H. Wu)

condition. The two conditions are verified numerically by real-life adaptive meshes (see, e.g., [25,26]). The results in [25] have been applied to enhance the eigenvalue approximations by the finite element method (see [26]).

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain with boundary $\partial\Omega$. Let $\mathscr{M}_h$ be a regular triangulation of Ω, $\mathscr{E}_h$ be the set of all interior edges, and $\mathscr{N}_h$ be the set of all nodal points. Assume that the origin $O \in \mathscr{N}_h$ and any triangle $\tau \in \mathscr{M}_h$ is considered as closed. Let $V_h^k = \left\{v_h : v_h \in H^1(\Omega), v_h|_\tau \in P_k(\tau), \forall \tau \mathscr{M}_h\right\}$ be the conforming finite element space associated with $\mathscr{M}_h$. Here P_k denotes the set of polynomials with degree $\leq k$. We remark that we will use the total degrees of freedom N (instead of the maximum mesh size h) to measure the rate of convergence. However, for notational convenience, we are still using h as an index.

Given a continuous function φ, the recovered gradient by PPR, $G_h\varphi$, is a vector function in $V_h^k \times V_h^k$ that is defined as follows [18,19]. For a node $z \in \mathscr{N}_h$, we select $n \geq (k+2)(k+3)/2$ sampling points $z_j \in \mathscr{N}_h$, $j = 1,2,\cdots,n$, in an element patch ω_z containing z (z is one of z_j), and fit a polynomial of degree $k+1$, in the least squares sense, with values of φ at those sampling points. In other words, we are looking for $p_{k+1} \in \mathscr{P}_{k+1}$ such that

$$\sum_{j=1}^{n}(p_{k+1}-\varphi)^2(z_j) = \min_{q\in\mathscr{P}_{k+1}} \sum_{j=1}^{n}(q-\varphi)^2(z_j). \tag{1.1}$$

The recovered gradient at z is then defined as

$$G_h\varphi(z) = (\nabla p_{k+1})(z). \tag{1.2}$$

Suppose u is an unknown solution and $u_h \in V_h^k$ is an approximation of u. If $G_h u_h$ is a better approximation than ∇u_h, then we can use $\left\|G_h u_h - \nabla u_h\right\|$ as an a posteriori error estimate of $\left\|\nabla u - \nabla u_h\right\|$.

Note that the recovered gradient by PPR is constructed node-wise. In the case of adaptive refined meshes, the number of elements surround a node may varies for different nodes. This makes the implementation of PPR on adaptive meshes a little complicated, in particular dealing with the boundary nodes. Inspiring by the fact that the local a posteriori indicators are usually calculated element-wise, we introduce a modified polynomial preserving gradient recovery which is construct element-wise. That is, for any element $\tau \in \mathscr{M}_h$, we choose sampling points from the nodes in an element patch ω_τ containing τ and construct recovered gradient on τ. For interior element τ, the element patch ω_τ can be chosen as the union of τ and the three elements that have common edges with τ. Note that ω_τ has the same topology structure for interior elements. We refer to Fig. 1 for some possible choices of sampling points. For an element τ whose one edge is on the boundary of Ω, a simple choice of ω_τ is $\omega_{\tau'}$, where τ' is an interior element that has a common edge with the element τ.

In this paper, we extend the results in [25] and [26] on PPR to the case of the modified polynomial preserving recovery (MPPR). We first consider the application of recovered gradient by MPPR to adaptive finite element methods for elliptic problems. The superconvergence of the recovered gradient by MPPR and the exactness of the a posteriori error estimate based on MPPR are proved for the Poisson's equation under the two mesh conditions