Numer. Math. Theor. Meth. Appl.
doi: 10.4208/nmtma.2010.32s.2

Vol. **3**, No. 2, pp. 143-161
May 2010

Analysis of a Class of Symmetric Equilibrium Configurations for a Territorial Model

Michael Busch* and Jeff Moehlis

Department of Mechanical Engineering, University of California, Santa Barbara, CA 93106, USA.

Received 15 September 2009; Accepted (in revised version) 6 January 2010

Available online 8 March 2010

Abstract. Motivated by an animal territoriality model, we consider a centroidal Voronoi tessellation algorithm from a dynamical systems perspective. In doing so, we discuss the stability of an aligned equilibrium configuration for a rectangular domain that exhibits interesting symmetry properties. We also demonstrate the procedure for performing a center manifold reduction on the system to extract a set of coordinates which capture the long term dynamics when the system is close to a bifurcation. Bifurcations of the system restricted to the center manifold are then classified and compared to numerical results. Although we analyze a specific set-up, these methods can in principle be applied to any bifurcation point of any equilibrium for any domain.

AMS subject classifications: 37N25, 37G10

Key words: Territorial behavior, Voronoi tessellations, bifurcation, center manifold reduction.

1. Introduction

A *territory* is a geographical area that an individual animal consistently defends against other individuals from its own species, typically in an attempt to maximize its reproductive opportunities and/or to secure food resources for itself and its young [11]. Territoriality is common across nearly all major groups of organisms on the planet. While higher animals like vertebrates exhibit the most obvious territorial boundaries, lower animals like invertebrates, plants, fungi and possibly even bacteria are known to aggressively defend space through behaviors and chemicals.

The recent paper [10] studied equilibrium configurations for a model for territorial behavior based on Voronoi tessellations, which captures interactions between agents in a simple way [9]; also see [3, 6, 7]. For this model, at a given time and for each agent, one calculates the set of points in the domain of interest which are closer to that agent than to any other. Such a partition of the domain is called a Voronoi tessellation, and the set of

*Corresponding author. *Email addresses:* `mbusch@engineering.ucsb.edu` (M. Busch), `moehlis@engineering.ucsb.edu` (J. Moehlis)

such points for each agent is called the agent's Voronoi cell. The agents then move toward the centroid of their Voronoi cell, continuing such adjustment until an equilibrium state is reached. An agent's Voronoi cell at such an equilibrium is considered to be its territory. We note that these equilibria are centroidal Voronoi tessellations, that is, Voronoi tessellations for which the generators of the Voronoi cells are the centroids of the cells defined using a constant density function [3]. This model captures the tendency of each agent to occupy territory so that it is as far from others as possible, and the notion that aggression of an agent decreases monotonically with distance from the center of its territory. It ignores environmental influences and heterogeneity in the individuals' characteristics or behavior. We remark that related models based on Voronoi tessellations have become popular in the robotics literature, e.g. [2, 5, 8], where the motivation might be the performance of spatially distributed sensing tasks such as surveillance or search and rescue.

This model was considered in detail in [10] for rectangular domains, with the boundaries of the domain forming boundaries of the Voronoi cells as appropriate. The analysis included a numerical bifurcation analysis for which the ratio L of the length of the shorter side to the length of the longer side of the rectangle was treated as a bifurcation parameter. This showed numerically how equilibrium configurations are related to each other through bifurcations, and identified ranges of L for which coexisting stable equilibrium configurations occur.

For a rectangular domain with any L, there are equilibrium configurations in which all agents are aligned. As shown in [10], such configurations can be stable or unstable depending on the number of agents and the value of L, and they can undergo bifurcations as L is varied. In this paper, we consider the stability and bifurcation behavior of these configurations analytically. In particular, in Sections 2 and 3 we respectively specify the territorial model and the aligned equilibrium configurations more precisely. In Section 4 we calculate the eigenvalues and eigenvectors associated with the linearization about such equilibrium configurations, and in Section 5 we identify parameter values at which bifurcations occur. In Section 6 we perform a center manifold reduction for small numbers of agents, which captures the asymptotic dynamics of the system near the bifurcation. The analysis of the dynamics reduced to the center manifold allows a classification of the bifurcation which occurs for the system. For a system with two agents, the results from the center manifold analysis are shown to be consistent with results from an alternative treatment given in the Appendix. Finally, Section 8 gives concluding remarks.

2. Model description

The model studied in [10] considers N agents in a two-dimensional rectangular domain D with sides of length 1 and L. The location of the i^{th} agent at time step n is $\mathbf{x}_i^{(n)}$. The Voronoi cell [3] for the i^{th} agent at time step n is defined as

$$V_i^{(n)} = \left\{ \mathbf{x} \in D \mid |\mathbf{x} - \mathbf{x}_i^{(n)}| < |\mathbf{x} - \mathbf{x}_j^{(n)}| \quad \text{for } j = 1, \cdots, N, \ j \neq i \right\},$$