Numer. Math. Theor. Meth. Appl.
doi: 10.4208/nmtma.2010.32s.4

Vol. **3**, No. 2, pp. 178-194
May 2010

Three-Dimensional Finite Element Superconvergent Gradient Recovery on Par6 Patterns

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Received 15 September 2009; Accepted (in revised version) 6 January 2010

Available online 8 March 2010

Abstract. In this paper, we present a theoretical analysis for linear finite element superconvergent gradient recovery on Par6 mesh, the dual of which is centroidal Voronoi tessellations with the lowest energy per unit volume and is the congruent cell predicted by the three-dimensional Gersho's conjecture. We show that the linear finite element solution u_h and the linear interpolation u_I have superclose gradient on Par6 meshes. Consequently, the gradient recovered from the finite element solution by using the superconvergence patch recovery method is superconvergent to ∇u. A numerical example is presented to verify the theoretical result.

AMS subject classifications: 65N30

Key words: Superconvergence, Par6, finite element method, centroidal Voronoi tessellations, Gersho's conjecture.

1. Introduction

Superconvergence of the gradient for the finite element approximation is a phenomenon whereby the convergent order of the derivatives of the finite element solutions exceeds the optimal global rate. Extensive research works have been done on this active research topic (see, e.g., Wahlbin [1], Křížek [2], Chen and Huang [3]). Various postprocessing techniques are raised to recover the gradients with high order accuracy from the finite element solution, such as the well-known Superconvergence Patch Recovery (SPR) method introduced by Zienkiewicz and Zhu [4] and Polynomial Patch Recovery (PPR) method raised by Zhang and Naga [5]. The superconvergence property in the gradient recovery has been applied to a posterior error estimation and mesh adaptivity with huge success, especially in numerical simulations in engineering [6–10].

The geometric structure of the computational mesh has great influence on the superconvergence property. It is well-known that superconvergence property is preserved for

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both the function values of finite element solution and its derivatives on equilateral meshes. However the requirement of an equilateral mesh is too stringent and usually it is impossible to generate such a mesh for a general domain, even for a simple rectangular domain. Thus some geometric conditions are imposed on computational meshes to guarantee the superconvergence property, such as the $\mathcal{O}(h^2)$ approximate parallelogram property by Xu and Bank [11, 12]. But questions such as what kind of meshes satisfy these conditions and how to generate these meshes are still unsolved. Recently, these questions are partially answered in [13, 14]. The superconvergence based on centroidal Voronoi tessellation (CVT) for linear finite element approximation is reported and based on this result a posterior error estimator and corresponding adaptive CVT-based mesh generation method are also raised. This finding is extended to three dimension by Chen, Huang and Wang [15] with a new recovery method: Modified Superconvergence Patch recovery (MSPR) method to overcome the influence of slivers. The theoretical proof for superconvergence property based on general CVT meshes seems quite difficult.

We take the initial step by giving the theoretical analysis of superconvergence property on a particular CVT structure: Par6 tessellation. The details of Par6 are presented in Section 2. Following earlier works [16, 17], we first present the result that the gradient of the linear finite element approximation u_h is superconvergent to the gradient of the piecewise linear interpolant u_I of the solution u. More precisely, we have

$$\left\|u_h - u_I\right\|_{1,\Omega} \lesssim h^2 \|u\|_{3,\infty,\Omega}.$$

Here the convergence order is approximately $\frac{1}{2}$ higher than general cases on general CVT meshes because of the highly symmetric structure of Par6 tessellation. The low order terms in the asymptotic expansion of the local error are totally canceled on Par6 structure. The errors on the boundary are also treated in this estimation. The second major part of this proof is that the gradient recovered by SPR is superconvergent to true gradient, that is to say

$$\left\|\nabla u - G_h u_h\right\|_{0,\Omega} \lesssim h^2 \|u\|_{3,\infty,\Omega},$$

where G_h is the SPR recovery operator. Both the superconvergence and the gradient recovery results are for a non-self-adjoint and possibly indefinite problem.

The rest of this paper is organized as follows. A detailed introduction of Par6 tessellation is given in Section 2. The theoretical proof of superconvergence property on Par6 tessellation is presented in Section 3. And in Section 4, numerical example is presented to verify the theoretical result. Finally, some conclusions are drawn in Section 5.

2. Par6 pattern: the optimal centroidal Voronoi tessellation in three dimensional space

Par6 pattern is a assembly which can be repeated indefinitely to fill space [18]. Different from two dimensional case, regular tetrahedrons, unlike equilateral triangles, can not be fitted together to fill space. Par6 assembly is obtained by distorting a cube into a