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Fast Solvers of Fredholm Optimal Control Problems

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Abstract. The formulation of optimal control problems governed by Fredholm integral equations of second kind and an efficient computational framework for solving these control problems is presented. Existence and uniqueness of optimal solutions is proved. A collective Gauss-Seidel scheme and a multigrid scheme are discussed. Optimal computational performance of these iterative schemes is proved by local Fourier analysis and demonstrated by results of numerical experiments.

AMS subject classifications: 49K22, 65K10, 65R20

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1. Introduction

Fast iterative methods and optimization related to differential and integral equations are two important fields of research in applied mathematics. The purpose of optimization is to define ways of how optimally change or influence real world systems to meet a given target. This requires to realize large-scale optimization strategies with increasing complexity that in turn motivates the development of fast iterative schemes for optimization purposes.

We focus on the optimization framework provided by infinite-dimensional optimal control theory as pioneered in [12] with partial differential equations. In this framework, we consider a governing state equation, a description of the control mechanism, and a criterion defining the objective that models the purpose of the control and describes the cost of its action. An optimal control problem is then formulated as the minimization of the objective under the constraint given by the modeling equations.

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While optimization with partial differential equations (PDE) has received much attention [3, 9], much less is known on optimization problems with integro-differential equations and with integral equations. The purpose of this paper is to discuss the formulation of an optimal control of a system governed by a Fredholm integral equation of second kind and to present two iterative schemes that solve the corresponding optimality system with optimal computational complexity.

Also from the scientific computing point of view, the numerical solution of Fredholm integral equations is a well established mathematical field [1,7], while much less is known on efficient solution procedures of related optimal control problems. We present onegrid and multigrid iterative schemes that solve linear distributed optimal control problems governed by Fredholm integral equations. We prove mesh-independent convergence of these scheme and robustness with respect to the value of optimization parameters.

Fredholm integral equations of second kind arise naturally in the theory of signal processing [10,14] and in inverse problems [6]. They also play a main role in the modeling of thin wires antennas [16]. Optimal control of systems governed by integral equations are important in applications. In particular, we consider an optimal control problem related to the Ornstein-Uhlenbeck process that arises from statistical communication theory [10].

In the next section, a class of optimal control problems governed by Fredholm integral equations of second kind is formulated and existence and uniqueness of optimal solution is proved. In Section 3, a collective Gauss-Seidel scheme and a linear multigrid scheme are presented. The convergence properties of these iterative schemes are discussed in Section 4 in the framework of one-grid and two-grid local Fourier analysis. We show that the proposed iterative schemes are efficient and robust with respect to changes of the value of the optimization parameters. Specifically, using local Fourier analysis we obtain multigrid convergence factors that are mesh independent and these factors improve as the weight of the cost of the control becomes smaller. In Section 5, results of numerical experiments are reported that demonstrate optimal computational complexity and robustness of the proposed iterative solvers. These results appear in sharp agreement with the estimates obtained by Fourier analysis. A section of conclusion completes this work.

2. Optimal control with Fredholm integral equations of the second kind

We consider Fredholm integral equations of the second kind with linear distributed control mechanism. The purpose of the control is to determine a control function such that the resulting state $y \in L^2(\Omega)$ tracks as close as possible a desired target configuration $z \in L^2(\Omega)$ where Ω is the domain. The corresponding optimal control problem is formulated as the minimization of a cost functional J subject to the constraint given by an integral equation. We have

$$\min_{u \in L^2(\Omega)} J(y,u) := \frac{1}{2}\|y-z\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)}, \tag{2.1}$$

$$y = f(y) + u + g \qquad \text{in } \Omega. \tag{2.2}$$