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Some Properties of the Optimal Preconditioner and the Generalized Superoptimal Preconditioner

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Abstract. The optimal preconditioner and the superoptimal preconditioner were proposed in 1988 and 1992 respectively. They have been studied widely since then. Recently, Chen and Jin [6] extend the superoptimal preconditioner to a more general case by using the Moore-Penrose inverse. In this paper, we further study some useful properties of the optimal and the generalized superoptimal preconditioners. Several existing results are extended and new properties are developed.

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1. Introduction

Given a unitary matrix $U \in \mathbb{C}^{n\times n}$, let

$$\mathscr{M}_U \equiv \left\{U^*\Lambda_n U \mid \Lambda_n \text{ is any } n\text{-by-}n \text{ diagonal matrix}\right\}. \tag{1.1}$$

The optimal preconditioner $c_U(A_n)$ is defined to be the minimizer of

$$\min_{W_n\in\mathscr{M}_U} \left\|A_n - W_n\right\|_F .$$

This preconditioner was first proposed in [5] and then extended in [3, 12]. Due to be very efficient for solving a large class of structured systems [2, 4, 13, 14], the optimal preconditioner $c_U(A_n)$ has been studied deeply and widely. Many useful properties of $c_U(A_n)$ have been found.

Besides using the minimizer of $\min_{W_n} \left\|A_n - W_n\right\|_F$ as a preconditioner, Tyrtyshnikov [17] proposed another preconditioner $t_U(A_n)$, called superoptimal preconditioner, which is defined to be the minimizer of

$$\min_{W_n} \|I - W_n^{-1}A_n\|_F,$$

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where W_n runs over all nonsingular matrices in $\mathscr{M}_U$ defined as in (1.1), I denotes the identity matrix. Recent results demonstrate that the superoptimal preconditioner has good filtering capabilities when applied in signal/image processing [8,9].

Very recently, the definition of the superoptimal preconditioner is generalized by Chen and Jin [6] by using the Moore-Penrose inverse [1]. For any arbitrary matrix A_n, the generalized superoptimal preconditioner $t_U(A_n)$ is defined to be the minimizer of

$$\min_{W_n \in \mathscr{M}_U} \|I - W_n^{\dagger} A_n\|_F, \tag{1.2}$$

where $W_n^{\dagger}$ denotes the Moore-Penrose inverse of W_n. In [6], the authors give an explicit formula for this generalized superoptimal preconditioner and discuss its stability properties.

In this paper, we further study the optimal preconditioner and the generalized superoptimal preconditioner defined as in (1.2). The rest part of the paper is arranged as follows. In Section 2, we extend some existing results and develop some new properties of $c_U(A_n)$. In Section 3, the properties of $t_U(A_n)$ are discussed. The relation between the singular values of the optimal preconditioned matrix $c_U(A_n)^{\dagger}A_n$ and the superoptimal preconditioned matrix $t_U(A_n)^{\dagger}A_n$ is given in Section 4. Here, $c_U(A_n)^{\dagger} \equiv (c_U(A_n))^{\dagger}$ and $t_U(A_n)^{\dagger} \equiv (t_U(A_n))^{\dagger}$. Our results generalize some results presented in [15,16].

2. The optimal preconditioner $c_U(A_n)$

In this section, we discuss the properties of the optimal preconditioner $c_U(A_n)$. Let $\delta(E_n)$ denote the diagonal matrix whose diagonal is equal to the diagonal of the matrix E_n. We first introduce some lemmas and theorems which will be used later.

Lemma 2.1. (Lemma 3.5 in [14]; Theorem 1 in [16]) *Let $A_n \in \mathbb{C}^{n\times n}$ with $n \geq 1$ and U be any unitary matrix. Then*

(i) $c_U(A_n) \equiv U^*\delta(UA_nU^*)U$ *which is uniquely determined by* A_n.

(ii) $c_U(A_n^*) = c_U(A_n)^*$.

(iii) $c_U(B_nA_n) = B_n c_U(A_n)$, $c_U(A_nB_n) = c_U(A_n)B_n$, *for any* $B_n \in \mathscr{M}_U$.

Lemma 2.2. *Let $A_n \in \mathbb{C}^{n\times n}$ be partitioned as*

$$A_n = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad and \quad D_n = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}.$$

Then for any unitarily invariant norm $\|\cdot\|$, we have

$$\left\|D_n\right\| \leq \left\|A_n\right\|.$$