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Alternating Direction Finite Volume Element Methods for Three-Dimensional Parabolic Equations

Tongke Wang*

School of Mathematical Sciences, Tianjin Normal University, Tianjin 300387, China.

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Abstract. This paper presents alternating direction finite volume element methods for three-dimensional parabolic partial differential equations and gives four computational schemes, one is analogous to Douglas finite difference scheme with second-order splitting error, the other two schemes have third-order splitting error, and the last one is an extended LOD scheme. The L^2 norm and H^1 semi-norm error estimates are obtained for the first scheme and second one, respectively. Finally, two numerical examples are provided to illustrate the efficiency and accuracy of the methods.

AMS subject classifications: 65M08, 65M12, 65M15

Key words: Three-dimensional parabolic equation, alternating direction method, finite volume element method, error estimate.

1. Introduction

Finite volume element methods (FVEMs) [1–3] or generalized difference methods [4] discretize the integral form of conservation laws of differential equation by choosing linear or high order finite element space as the trial space. The method lies in between finite element method and finite difference method in concept and implementation. In recent years, some literature focused on the error estimates of finite volume element methods, especially for two dimensional problems, see the references [5–13]. Recently, the author [14] combines finite volume element methods and alternating direction methods for two dimensional parabolic differential equations and presents some alternating direction finite volume element schemes. Here, we further extend the method to three-dimensional partial differential equations. As an efficient technique, alternating direction method [15,16] successfully converts multidimensional problems to a collection of one dimensional problems, which can be solved very easily. Because ADI finite difference methods and alternating direction finite element methods are unconditionally stable and highly efficient, they have been applied in many areas of applied sciences [17,18]. It is worth mentioning that

*Corresponding author. *Email address:* wangtke@sina.com (T. Wang)

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Professor Douglas et al. [16] presented an LOD finite difference scheme with third-order perturbation term. In this paper, we write the finite volume element method as tensor product form by perturbing the differential equations, so we can convert the method to a series of one dimensional problems. We give four kinds of alternating direction schemes, the first one is similar to Douglas scheme [15] with second-order splitting error, the second and third are also Douglas schemes with third order splitting error [16]. The last one is an extended locally one dimensional (LOD) scheme [19]. It is worth mentioning that the LOD scheme in this paper completely decomposes multidimensional problems to a collection of one dimensional problems and the method is valid for nonhomogeneous differential equations with nonhomogeneous boundary conditions.

The remainder of the article is outlined as follows. In Section 2, we obtain a class of finite volume element method with tensor product form by perturbing the differential equation. We present four kinds of computation schemes. In Section 3, taking the fist scheme and the second one as two examples, we further analyze these schemes. By defining discrete L^2 norm and H^1 semi-norm, we obtain L^2 norm and H^1 semi-norm error estimates for the first scheme and the second one. Finally, in Section 4, we provides two numerical examples to illustrate the effectiveness of the four schemes.

Throughout the article C will denote a generic (sometimes large) constant and ϵ a generic small one independent of mesh-size h, where C and ϵ can have different values in different places.

2. Alternating direction FVEM for 3D parabolic equations

Consider the following three-dimensional parabolic problem on domain $\Omega=[0,1]^3$

$$\frac{\partial u}{\partial t}-\Delta u=f(x,y,z,t),\quad (x,y,z)\in\Omega,\quad t\in(0,T], \tag{2.1}$$

$$u|_{\partial\Omega}=0,\quad u(x,y,z,0)=u_0(x,y,z), \tag{2.2}$$

where $f(x,y,z,t)$ is sufficiently smooth.

First, give a cuboidal partition Q_h for Ω and the nodes are denoted by (x_i,y_j,z_k), $i(j,k)=0,1,\cdots,N_x(N_y,N_z)$. Let

$$h_i^x=x_i-x_{i-1},\quad h_j^y=y_j-y_{j-1},\quad h_k^z=z_k-z_{k-1},$$
$$h_x=\max_{1\le i\le N_x}h_i^x,\quad h_y=\max_{1\le j\le N_y}h_j^y,\quad h_z=\max_{1\le k\le N_z}h_k^z,\quad h=\max(h_x,h_y,h_z).$$

Further let

$$x_{i-\frac12}=x_i-\frac12 h_i^x,\quad x_{i+\frac12}=x_i+\frac12 h_{i+1}^x,$$
$$y_{j-\frac12}=y_j-\frac12 h_j^y,\quad y_{j+\frac12}=y_j+\frac12 h_{j+1}^y,$$
$$z_{k-\frac12}=z_k-\frac12 h_k^z,\quad z_{k+\frac12}=z_k+\frac12 h_{k+1}^z.$$