Numer. Math. Theor. Meth. Appl. Vol. **4**, No. 1, pp. 1-12
doi: 10.4208/nmtma.2011.m1003 February 2011

REVIEW ARTICLE

Weakly Admissible Meshes and Discrete Extremal Sets

Len Bos[1], Stefano De Marchi[2,*], Alvise Sommariva[2] and Marco Vianello[2]

[1] *Department of Computer Science, University of Verona, Strada Le Grazie 15 - 37134 Verona, Italy.*
[2] *Department of Pure and Applied Mathematics, University of Padua, Via Trieste 63 - 35121 Padua, Italy.*

Received 8 March 2010; Accepted (in revised version) 8 June 2010

Available online 29 October 2010

Abstract. We present a brief survey on (Weakly) Admissible Meshes and corresponding Discrete Extremal Sets, namely Approximate Fekete Points and Discrete Leja Points. These provide new computational tools for polynomial least squares and interpolation on multidimensional compact sets, with different applications such as numerical cubature, digital filtering, spectral and high-order methods for PDEs.

AMS subject classifications: 65D05, 65D32

Key words: Weakly admissible meshes, Approximate Fekete points, Discrete Leja points.

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1. Introduction

Locating good points for multivariate polynomial approximation, in particular interpolation, is an open challenging problem, even in standard domains. One set of points that is always good, in theory, is the so-called *Fekete points*. They are defined to be those points that maximize the (absolute value of the) Vandermonde determinant on the given compact set. However, these are known analytically in only a few instances (the interval and the complex circle for univariate interpolation, the cube for tensor product interpolation), and are very difficult to compute, requiring an expensive and numerically challenging nonlinear multivariate optimization.

Recently, a new insight has been given by the theory of "Admissible Meshes" of Calvi and Levenberg [9], which are nearly optimal for least-squares approximation and contain interpolation sets (Discrete Extremal Sets) nearly as good as Fekete points of the domain. Such sets, termed Approximate Fekete Points and Discrete Leja Points, are computed using

*Corresponding author. *Email addresses:* `leonardpeter.bos@univr.it` (L. Bos), `demarchi@math.unipd.it` (S. De Marchi), `alvise@math.unipd.it` (A. Sommariva), `marcov@math.unipd.it` (M. Vianello)

only basic tools of numerical linear algebra, namely QR and LU factorizations of Vandermonde matrices. Admissible Meshes and Discrete Extremal Sets allow us to replace a continuous compact set by a discrete version, that is "just as good" for all practical purposes.

2. Weakly Admissible Meshes (WAMs)

Given a *polynomial determining* compact set $K \subset \mathbb{R}^d$ or $K \subset \mathbb{C}^d$ (i.e., polynomials vanishing there are identically zero), a Weakly Admissible Mesh (WAM) is defined in [9] to be a sequence of discrete subsets $\mathcal{A}_n \subset K$ such that

$$\|p\|_K \leq C(\mathcal{A}_n)\|p\|_{\mathcal{A}_n}, \ \ \forall p \in \mathbb{P}_n^d(K), \tag{2.1}$$

$\mathbb{P}_n^d(K)$ being the set of d-variate polynomials of degree at most n on K, where both $\mathrm{card}(\mathcal{A}_n) \geq N := \dim(\mathbb{P}_n^d(K))$ and $C(\mathcal{A}_n)$ grow at most *polynomially* with n. When $C(\mathcal{A}_n)$ is bounded we speak of an Admissible Mesh (AM). Here and below, we use the notation $\|f\|_X = \sup_{x\in X} |f(x)|$, where f is a bounded function on the compact X.

We sketch below the main features of WAMs in terms of ten properties (cf. [4, 9]):

P1: $C(\mathcal{A}_n)$ is invariant under affine mapping

P2: any sequence of unisolvent interpolation sets whose Lebesgue constant grows at most polynomially with n is a WAM, $C(\mathcal{A}_n)$ being the Lebesgue constant itself

P3: any sequence of supersets of a WAM whose cardinalities grow polynomially with n is a WAM with the same constant $C(\mathcal{A}_n)$

P4: a finite union of WAMs is a WAM for the corresponding union of compacts, $C(\mathcal{A}_n)$ being the maximum of the corresponding constants

P5: a finite cartesian product of WAMs is a WAM for the corresponding product of compacts, $C(\mathcal{A}_n)$ being the product of the corresponding constants

P6: in $\mathbb{C}^d$ a WAM of the boundary ∂K is a WAM of K (by the maximum principle)

P7: given a polynomial mapping π_s of degree s, then $\pi_s(\mathcal{A}_{ns})$ is a WAM for $\pi_s(K)$ with constants $C(\mathcal{A}_{ns})$ (cf. [4, Prop.2])

P8: any K satisfying a Markov polynomial inequality like $\|\nabla p\|_K \leq Mn^r\|p\|_K$ has an AM with $\mathcal{O}(n^{rd})$ points (cf. [9, Thm.5])

P9: least-squares polynomial approximation of $f \in C(K)$: the least-squares polynomial $\mathcal{L}_{\mathcal{A}_n}f$ on a WAM is such that

$$\|f - \mathcal{L}_{\mathcal{A}_n}f\|_K \lessapprox C(\mathcal{A}_n)\sqrt{\mathrm{card}(\mathcal{A}_n)}\,\min\{\|f-p\|_K, p \in \mathbb{P}_n^d(K)\}$$

(cf. [9, Thm.1])

P10: Fekete points: the Lebesgue constant of Fekete points extracted from a WAM can be bounded like $\Lambda_n \leq NC(\mathcal{A}_n)$ (that is the elementary classical bound of the continuum Fekete points times a factor $C(\mathcal{A}_n)$); moreover, their asymptotic distribution is the same of the continuum Fekete points, in the sense that the corresponding discrete probability measures converge weak-$*$ to the pluripotential equilibrium measure of K (cf. [4, Thm.1]).