Numer. Math. Theor. Meth. Appl. Vol. **4**, No. 1, pp. 53-67
doi: 10.4208/nmtma.2011.m99009 February 2011

On Newton's Method for Solving Nonlinear Equations and Function Splitting

Ioannis K. Argyros[1,*] and Saïd Hilout[2]

[1] *Cameron University, Department of Mathematics Sciences, Lawton, OK 73505, USA.*
[2] *Université de Poitiers, Laboratoire de Mathématiques et Applications, Bd. Pierre et Marie Curie, Téléport 2, B.P. 30179, 86962 Futuroscope Chasseneuil Cedex, France.*

Received 25 February 2009; Accepted (in revised version) 18 October 2009

Available online 29 October 2010

Abstract. We provided in [14] and [15] a semilocal convergence analysis for Newton's method on a Banach space setting, by splitting the given operator. In this study, we improve the error bounds, order of convergence, and simplify the sufficient convergence conditions. Our results compare favorably with the Newton-Kantorovich theorem for solving equations.

AMS subject classifications: 65H99, 65H10, 65G99, 49M15, 47J20, 47H04, 90C30, 90C33

Key words: Newton's method, Banach space, majorizing sequence, semilocal convergence, splitting of an operator.

1. Introduction

Let $\mathscr{X}$ and $\mathscr{Y}$ be Banach spaces, let $\mathscr{D}$ be an open convex subset of $\mathscr{X}$, and let $F : \mathscr{D} \longmapsto \mathscr{Y}$ be a Fréchet differentiable function, fixed all throughout this paper. In the sequel, we will assume that the function F has a splitting

$$F = f + g, \tag{1.1}$$

where $f, g : \mathscr{D} \longmapsto \mathscr{Y}$ are Fréchet differentiable functions satisfying the condition

$$F'(x) = F'(u_0),\ \forall x \in \mathscr{D} \implies f'(x) = f'(u_0),\ \forall x \in \mathscr{D}. \tag{1.2}$$

What this means is that the splitting functions f and g should only be nonlinear if the initial function F is nonlinear. Given any $u_0 \in \mathscr{D}$ and $r > 0$, $\overline{U}(u_0, r)$ will designate the set $\{x \in \mathscr{X} : \|x - u_0\| \le r\}$, and $U(u_0, r)$ the corresponding interior ball.

*Corresponding author. *Email addresses:* `ioannisa@cameron.edu` (I. K. Argyros), `said.hilout@math.univ-poitiers.fr` (S. Hilout)

We are interested in the solvability of the equation

$$F(u) = 0. \tag{1.3}$$

A large number of problems in applied mathematics and also in engineering are solved by finding the solutions of certain equations. For example, dynamic systems are mathematically modeled by difference or differential equations, and their solutions usually represent the states of the systems. For the sake of simplicity, assume that a time-invariant system is driven by the equation $\dot{x} = Q(x)$ (for some suitable operator Q), where x is the state. Then the equilibrium states are determined by solving equation (1.3). Similar equations are used in the case of discrete systems. The unknowns of engineering equations can be functions (difference, differential, and integral equations), vectors (systems of linear or nonlinear algebraic equations), or real or complex numbers (single algebraic equations with single unknowns). Except in special cases, the most commonly used solution methods are iterative — when starting from one or several initial approximations a sequence is constructed that converges to a solution of the equation. Iteration methods are also applied for solving optimization problems. In such cases, the iteration sequences converge to an optimal solution of the problem at hand. Since all of these methods have the same recursive structure, they can be introduced and discussed in a general framework.

In previous papers co-authored by the first author, see [14, 15], we have provided (see also Theorems 2.2, 2.3, 3.1 and 3.2) an analysis under weaker sufficient convergent conditions than the celebrated Newton-Kantorovich theorem for solving equations (see Theorem 2.1).

Here, we improve the bounds on the distances $\| x_{k+1} - x_k \|$, $\| x_k - x^\star \|$, $(k \geq 0)$, and also simplify the sufficient convergence conditions for Newton method (2.1).

2. Preliminaries

In using the Newton's method

$$u_{m+1} = u_m - F'(u_m)^{-1} F(u_m), \tag{2.1}$$

one of the most important theorems in nonlinear analysis is the following result due — essentially — to Kantorovich [7].

Theorem 2.1 (The Kantorovich theorem). *Suppose that $F'(u_0)^{-1}$ exists for some $u_0 \in \mathscr{D}$, and that there exists $K \geq 0$ and $\eta \geq 0$ such that*

$$\| F'(u_0)^{-1} F(u_0) \| \leq \eta, \tag{2.2}$$

$$\| F'(u_0)^{-1} (F'(x) - F'(y)) \| \leq K \ \| x - y \|, \quad \forall x, y \in \mathscr{D}, \tag{2.3}$$

$$2 K \eta \leq 1. \tag{2.4}$$

Let

$$t_\star = \frac{2\,\eta}{1 + \sqrt{1 - 2\,K\,\eta}}, \qquad T_\star = \frac{2}{K} - t_\star,$$