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Spectral Deferred Correction Methods for Fractional Differential Equations

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Dedicated to Professor Xiaoqing Jin on the occasion of his 60th birthday

Abstract. In this paper, we propose and analyze a spectral deferred correction method for the fractional differential equation of order α. The proposed method is based on a well-known finite difference method of $(2-\alpha)$-order, see [Sun and Wu, Appl. Numer. Math., 56(2), 2006] and [Lin and Xu, J. Comput. Phys., 225(2), 2007], for prediction of the numerical solution, which is then corrected through a spectral deferred correction method. In order to derive the convergence rate of the prediction-correction iteration, we first derive an error estimate for the $(2-\alpha)$-order finite difference method on some non-uniform meshes. Then the convergence rate of orders $\mathscr{O}\left(\tau^{(2-\alpha)(p+1)}\right)$ and $\mathscr{O}\left(\tau^{(2-\alpha)+p}\right)$ of the overall scheme is demonstrated numerically for the uniform mesh and the Gauss-Lobatto mesh respectively, where τ is the maximal time step size and p is the number of correction steps. The performed numerical test confirms the efficiency of the proposed method.

AMS subject classifications: 26A33, 65M70, 34A08, 45K05

Key words: Fractional differential equation, spectral deferred correction method, finite difference method.

1. Introduction

In recent years, fractional differential equations have been attracting increasing attention as they have been applied to diverse fields, including control theory, biology, electrochemical processes, porous media, viscoelastic materials, polymer, finance, and etc; see, e.g., [1, 2, 5–7, 16, 19, 26, 28, 35, 37, 38] and the references therein.

As one of the basic fractional partial differential equations, the fractional ordinary differential equations (FODEs) of the form $\partial_t^\alpha u(t) = f(t, u(t))$ considered in this paper is

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of importance not only in its own right, but also in the fact that it reflexes the main feature and difficulty of more general fractional equations. This equation can be derived by using continuous time random walks [18, 36], in which the fractional derivative is used to represent the degree of memory in the particle spreading. Some effort has been made in constructing the exact solution of FODEs; see, e.g., the solution expression of more general integro-differential equations by using the Mittag-Leffler function [25], the Adomian decomposition analytical method to solve linear FODEs [22], homotopy methods [3, 21], a theoretical survey of FODEs [12], etc. A number of numerical methods have been proposed to discretize the fractional derivative, see, e.g., Liu et al. [33] for a finite difference method for the fractional diffusion equation, Langlands and Henry [29] for the L1 scheme, Sun and Wu [41] for finite difference method of the fractional diffusion-wave equation, Lin and Xu [32] for a rigorous convergence analysis of the L1 scheme, Deng [11] for the time fractional Fokker-Planck equation, Sun et al. [40] for an alternating direction implicit scheme, Diethelm et al. [13, 14] for a predictor-corrector schema and a fractional Adams method, Kumar et al. [27] and Cao and Xu [8] for the block-by-block methods, Garrappa [17] for the Adams multistep method, Saadatmandi et al. [39] for an operational matrix method, Jin et al. [24] for an analysis of two semidiscrete schemes, Cao et al. [9] and Gao et al. [20] for a higher order scheme, Zhao and Deng [43] for a Jacobian-predictor-corrector method, and so on.

In this paper we aim at proposing and analyzing a spectral deferred correction (SDC) method for the fractional ordinary differential equations. The SDC method has been known as an efficient methodology for solving the classical (integer order) ordinary differential equations, which was originally introduced by Dutt et al. [15]. This method can achieve high order convergence through a well designed prediction-correction iteration. The first attempt to extend the SDC to FODEs was made by Xin et al. [42]. They constructed a spectral deferred correction scheme for a Volterra integral equation which is equivalent to the FODEs. The numerical results presented in [42] show that high accuracy can be achieved using relatively few nodes as compared to the fractional block-by-block method. However, there is no theoretical analysis available in the litterature. Another related work, done by Mao et al. [34], is the realisation of a semi-implicit SDC method in time for water wave models with nonlocal viscous term.

In the present work, we attempt to construct a SDC scheme for the FODEs, and carry out an analysis for the convergence rate of the proposed method. The new SDC method for the FODEs is based on a finite difference scheme for the prediction of the numerical solution. This finite difference scheme is a generalisation of the well-known $(2-\alpha)$-order scheme on the constant time step [32,33,41] to some non-uniform grid. In order to analyze the convergence rate of the SDC method, we first establish an error estimate for the FD scheme in the prediction step. This consists of an extension of the existing theoretical result obtained for the uniform grid (i.e., constant time step size) to some non-uniform grid of different types. Our analysis and numerical experiments show that the convergence order of the overall scheme increases by $(2-\alpha)$ and 1 respectively for the uniform grid and Gauss-Lobatto grid after each prediction-correction loop.

The rest of the paper is organized as follows: In the next section, we describe the SDC