Numer. Math. Theor. Meth. Appl. Vol. **11**, No. 4, pp. 854-876
doi: 10.4208/nmtma.2018.s10 November 2018

A Space-Time Petrov-Galerkin Spectral Method for Time Fractional Diffusion Equation

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Received 13 September 2017; Accepted (in revised version) 19 December 2017

Dedicated to Professor Xiaoqing Jin on the occasion of his 60th birthday

Abstract. We develop in this paper a space-time Petrov-Galerkin spectral method for linear and nonlinear time fractional diffusion equations (TFDEs) involving either a Caputo or Riemann-Liouville derivative. Our space-time spectral method are based on generalized Jacobi functions (GJFs) in time and Fourier-like basis functions in space. A complete error analysis is carried out for both linear and nonlinear TFDEs. Numerical experiments are presented to demonstrate the effectiveness of the proposed method.

AMS subject classifications: 26A33, 34A08, 42C10, 65N35, 65N15

Key words: Fractional differential operator, generalised Jacobi functions, Fourier-like basis function, space-time spectral method, error analysis.

1. Introduction

We consider in this paper the following time fractional diffusion equation

$$ {}_0^C D_t^\alpha u(x,t) - \gamma\Delta u(x,t) + \mathscr{N}(u(x,t),t) = 0, \qquad \forall (x,t)\in\Omega = (-1,1)\times(0,T] \tag{1.1}$$

with the initial and boundary conditions

$$ u(x,0) = u_0(x), \qquad \forall x\in(-1,1), \tag{1.2a}$$
$$ u(\pm 1,t) = 0, \qquad \forall t\in[0,T]. \tag{1.2b}$$

In the above, $\alpha\in(0,1)$, γ is a diffusion constant, $\mathscr{N}$ is a linear or nonlinear operator, and ${}_0^C D_t^\alpha$ refers to left-sided Caputo fractional derivative of order α (see the definition in (2.4)). For the sake of simplicity, we choose to concentrate on the case of one spatial

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dimension. However, the proposed method and analysis can be directly extended to the multi-dimensional case with rectangular domains thanks to the Fourier-like basis functions that we employ in this paper.

The TFDEs (1.1) are frequently used to model anomalous diffusion in heterogeneous media, and its numerical solution has attracted much attention recently, including [15, 22, 28, 29] which use finite difference methods, and [11–13, 26] which use finite element methods.

Two main difficulties in solving fractional PDEs such as (1.1) are (i) fractional derivatives are non-local operators and generally lead to full matrices; and (ii) their solutions are often singular so polynomial based approximations are not efficient.

Since spectral methods are capable of providing exceedingly accurate numerical results with less degrees of freedoms, they have been widely used for numerical approximations [1, 9, 19, 20]. In particular, well designed spectral methods appear to be particularly attractive to deal with the difficulties associated with fractional PDEs mentioned above. Polynomial based spectral methods have been developed for TFDEs, e.g., [3, 14, 23]. However, these methods use polynomial basis functions which are not particularly suitable for TFDEs whose solutions are generally non-smooth at $t = 0$. Zayernouri and Karniadakis [25] first proposed to approximate the singular solutions by Jacobi poly-fractonomials, which were defined as eigenfunctions of a fractional Sturm-Liouville problem. Chen, Shen and Wang [4] constructed efficient Petrov-Galerkin methods for fractional PDEs by using the generalized Jacobi functions (GJFs) which include Jacobi poly-fractonomials as special cases. Subsequently, some authors developed spectral methods by using nodal GJFs for fractional PDEs [10, 24, 27, 30]. Most of these work are concerned with linear equations only.

We recall that the method presented in [4] is very efficient and accurate for fractional differential equations in time of the kind:

$$ {}_0^C D_t^{\alpha} u(t) = f(t), \quad u(0) = u_0. $$

On the other hand, an efficient and accurate space-time spectral method based on Fourier-like basis functions is developed in [21] for solving linear or nonlinear parabolic equations. In this paper, we combine the approaches in [4] and [21] to construct an efficient space-time spectral method for solving TFDEs. We highlight below the main contributions of this paper.

- For the time variable, the choice of GJFs can be tuned to match the singularities of the underlying solutions; and for the spatial variables, we use the Fourier-like basis functions. This combination greatly improves the accuracy and simplifies the implementation.

- We carry out error analysis of the proposed method for linear and nonlinear cases.

The organization of this paper is as follows. In the next section, we introduce the basis functions in both time and spatial directions, present some useful properties of fractional