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Efficient Chebyshev Spectral Method for Solving Linear Elliptic PDEs Using Quasi-Inverse Technique

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Abstract. We present a systematic and efficient Chebyshev spectral method using quasi-inverse technique to directly solve the second order equation with the homogeneous Robin boundary conditions and the fourth order equation with the first and second boundary conditions. The key to the efficiency of the method is to multiply quasi-inverse matrix on both sides of discrete systems, which leads to band structure systems. We can obtain high order accuracy with less computational cost. For multi-dimensional and more complicated linear elliptic PDEs, the advantage of this methodology is obvious. Numerical results indicate that the spectral accuracy is achieved and the proposed method is very efficient for 2-D high order problems.

AMS subject classifications: 65N35, 65N22, 65F05, 35J05

Key words: Chebyshev spectral method, quasi-inverse, Helmholtz equation, Robin boundary conditions, general biharmonic equation.

1. Introduction

Due to high order accuracy, spectral methods have gained increasing popularity for several decades, especially in the field of computational fluid dynamics (see, e.g., [1, 2] and the references therein). According to different test functions in a variational formulation, there are three most common spectral schemes, namely, the collocation, Galerkin and tau methods. Since the collocation methods approximate differential equations in physical space, it is very easy to implement and adaptable to various of problems, including variable coefficient and nonlinear differential equations. Weideman and Reddy constructed a MATLAB software suit to solve differential equations by the spectral collocation methods in [13]. Trefethen's book [12] explained the essentials of spectral collocation methods with the aid of 40 short MATLAB programs. For multi-dimensional problems, the spectral collocation methods discretize the differential operators employing Kronecker products. In

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the Galerkin method, we work in the spectral space, it may lead to well conditioned linear systems with sparse matrices for problems with constant coefficients by choosing proper basis functions (see, e.g., [3, 5, 9, 10]).

Although the collocation and Galerkin methods usually lead to optimal error estimates, the primary drawback of collocation method is that the differentiation matrices are dense in all dimensions, and it is generally not feasible to solve multi-dimensional problems by employing the Galerkin method. Shen used a matrix diagonalization method to solve the 2-D and 3-D Helmholtz problems in [9] and [10], but an eigenvalue-eigenvector decomposition of the discretized linear operator is required. Therefore it can only be used for relatively simple differential equations. Heinrichs [6] utilized a Galerkin basis set to obtain efficient differentiation matrices, and exploited the inherent structure of both the Galerkin differentiation matrices and the relationship between the Chebyshev and Galerkin spectral coefficients to maximize the sparsity of differential operators. Julien and Watson [7] presented the quasi-inverse technique to efficiently solve linear elliptic differential equations with constant coefficients under Dirichlet boundary conditions. In this paper, we present an extension of the Chebyshev spectral method using quasi-inverse technique to directly solve the Helmholtz equation with the homogeneous Robin boundary conditions and the general biharmonic equation with the first and second boundary conditions. For the general biharmonic equation, we give a uniform treatment for the first and second boundary conditions.

The main idea is that we employ a truncated series of Chebyshev polynomials to approximate the unknown function, and the differential operator is expanded by Chebyshev polynomials which vector of coefficients is represented by the product of derivative matrix and vector of Chebyshev coefficients of unknown function. The coefficients of this series are taken to be equal to the coefficients of the right-hand side expansion. According to Galerkin basis satisfying boundary conditions, we identify a transformation matrix which transforms the Chebyshev and Galerkin coefficients, and then multiply a quasi-inverse matrix on both sides of the resulting spectral system to obtain a pre-multiplied system $A\bar{v} = B\bar{f}$, where A and B have band structure. After we solve this system of equations, the Galerkin spectral coefficients are converted back to Chebyshev spectral coefficients. We obtain the approximation solution from spectral space to physical space using the forward Chebyshev transform by FFT.

The remainder of the paper is organized as follows. In the next section, we introduce some notations and summarize a few mathematical facts used in the remainder of the paper. In Section 3, we consider the Helmholtz equations for one, two and three dimensional cases. In Section 4, we study the general biharmonic equations for one and two dimensional cases. In Section 5, we present some numerical results. Finally, some concluding remarks are given in Section 6.

2. Preliminaries

2.1. Notation

We introduce some basic notations as follows: