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An Efficient Numerical Method for the Quintic Complex Swift-Hohenberg Equation

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Abstract. In this paper, we present an efficient time-splitting Fourier spectral method for the quintic complex Swift-Hohenberg equation. Using the Strang time-splitting technique, we split the equation into linear part and nonlinear part. The linear part is solved with Fourier Pseudospectral method; the nonlinear part is solved analytically. We show that the method is easy to be applied and second-order in time and spectrally accurate in space. We apply the method to investigate soliton propagation, soliton interaction, and generation of stable moving pulses in one dimension and stable vortex solitons in two dimensions.

AMS subject classifications: 65M70, 65Z05

Key words: Quintic complex Swift-Hohenberg equation, time-splitting Fourier pseudospectral method, numerical simulation, soliton.

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1. Introduction

Swift-Hohenberg equation was proposed by J. Swift and P. C. Hohenberg in 1977 along with the arising interest in stable spatially localized states [18–20]. It has been viewed as a model equation for a large class of higher-order parabolic model equations arising in a wide range of applications, such as the extended Fisher-Kolmogorov equation in statistical mechanics and the perturbed diffusion equation in phase field models [9, 14].

It is undoubtedly convenient if one can use just a single equation to explain a complicated phenomena in various systems. Initially, a single quintic complex Ginzburg-Landau (QCGL) equation was known to be able to do this for a laser with a fast saturable absorber. Its quintic nonlinearity is essential to ensure the stability of soliton-like pulses overcoming something that the cubic Ginzburg-Landau equation could not achieve. However, this

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model is restricted to a second-order term and a spectral response with a single maximum, which is not the case in many experiments. In order to make the model more realistic. the addition of a fourth-order spectral filtering term into the QCGL equation is needed and this is how the quintic complex Swift-Hohenberg equation emerged.

The quintic complex Swift-Hohenberg equation has a wide range of applications. It has been used to study instabilities and pattern formation phenomena in cases of Rayleigh-Bernard convection [4] and oscillating chemical reactions. In optics, this equation has been considered in relation to spatial structures in large aspect lasers [19] and synchronously pumped optical parametric oscillators [20, 29]. It is also important for describing pulse generation processes in passively mode-locked lasers with fast saturable absorbers [30].

The problem which we study numerically is the quintic complex Swift-Hohenberg (QCSH) equation [15, 19, 20, 22–24, 28], expressed as follows

$$i\frac{\partial \psi}{\partial t} = a\psi + b|\psi|^2\psi + c|\psi|^4\psi + d\triangle\psi + f\triangle^2\psi, \qquad \mathbf{x} \in R^d,\ t \geq 0, \tag{1.1}$$

where the function $\psi = \psi(\mathbf{x}, t)$ is a complex wave function with respect to time variable t and space variable $\mathbf{x}$, and a, b, c, d, and f are complex. f is also usually called viscous term in the equation.

Eq. (1.1) covers many nonlinear equations arising in various applications. For example, when a, b, c, d, and f are pure imaginary, it is the real Swift-Hohenberg equation. It has been used to study the collection between subcritical square patterns and oscillons in vertically vibrated granular layers [10] and to describe a wide range of patterns from Ravleigh-Benard convection to wide-area lasers [4]. When $f = 0$, i.e. excluding the viscous term, it collapses to the QCGL equation, which has been studied in many areas, such as fluid dynamics [17], chemical oscillations [18], and nonlinear optics [1]. Furthermore, if all the parameters in the QCGL equation are pure imaginary, it becomes the real Ginzburg-Landau equation. It has been used to examine the interaction of fronts in localized traveling wave pulses in binary liquid mixtures [16]. On the contrary, if the parameters are all real, the equation reduces to the nonlinear Schrödinger equation, which has been used for modeling for example, superfluidity or Bose-Einstein condensation [7].

There has been a lot of studies conducted for the QCSH equation without the viscous term i.e., $f = 0$. For example, stable pulse solutions, namely plain pulse which has the standard soliton shape, narrow composite pulse, and wide composite Pulse, have been found numerically for a wide range of parameters well beyond the region where analytical solutions exist [3]. The interactions between the more complicated structure of the composite pulses have been studied, too. Through numerical simulation, collision of counter-propagating pulses and vortices was studied by Sakaguchi et al. [26]. The results showed that the acceleration occurs at the collisions of two counter-propagating one-dimensional pulse and two dimensional vortices. For a fixed set of parameters, it was observed that the moving pulses are accelerated and the energy increases at the moment of collision.

Unlike the QCSH equation described above, numerical simulations conducted specifically for the QCSH equation with the viscous term are still very limited. We review several numerical results with regards to this equation in the following.