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Convergence Estimates for Some Regularization Methods to Solve a Cauchy Problem of the Laplace Equation

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Abstract. In this paper, we give a general proof on convergence estimates for some regularization methods to solve a Cauchy problem for the Laplace equation in a rectangular domain. The regularization methods we considered are: a non-local boundary value problem method, a boundary Tikhonov regularization method and a generalized method. Based on the conditional stability estimates, the convergence estimates for various regularization methods are easily obtained under the simple verifications of some conditions. Numerical results for one example show that the proposed numerical methods are effective and stable.

AMS subject classifications: 65N12, 65N15, 65N20, 65N21

Key words: Cauchy problem, Laplace equation, regularization methods, convergence estimates.

1. Introduction

In this paper, we consider a Cauchy problem for the Laplace equation in a rectangular domain as follows

$$\Delta u(x,y)=0, \qquad 0<x<\pi, \quad 0<y<a, \tag{1.1}$$

$$u(0,y)=u(\pi,y)=0, \quad 0\le y\le a, \tag{1.2}$$

$$u_y(x,0)=0, \qquad 0\le x\le \pi, \tag{1.3}$$

$$u(x,0)=\varphi(x), \qquad 0\le x\le \pi, \tag{1.4}$$

where a is a positive constant.

Define the family of rectangular regions with parameter $0<\sigma\le a$ by

$$D_\sigma=\Big\{(x,y)\ \Big|\ 0<x<\pi, 0<y<\sigma\Big\}. \tag{1.5}$$

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Assume that the exact Dirichlet data $\varphi \in L^2(0,\pi)$ and the measured data $\varphi^\delta \in L^2(0,\pi)$ satisfy

$$\|\varphi^\delta - \varphi\| \le \delta, \tag{1.6}$$

where $\|\cdot\|$ denotes the L^2-norm and $\delta > 0$ is a noise level.

Further we assume that the following *a-priori* bound exists,

$$\|u(\cdot,a)\| \le E, \tag{1.7}$$

or a stronger *a-priori* bound assumption holds

$$\left\| \frac{\partial^p u}{\partial y^p}(\cdot,a) \right\| \le E_p, \tag{1.8}$$

where E and E_p are positive constants and $p \ge 1$ is an integer number.

The problem is to find an approximate solution for problem (1.1)–(1.4) with noisy data φ^δ instead of φ. This is a classical Cauchy problem for Laplace's equation in a special domain. It arises in many real applications, such as nondestructive testing [1, 7], geophysics [26] and cardiology [8]. It is well known that the problem is typically ill-posed. That is, any small changes of the Cauchy data may induce large changes of the solutions (e.g. [15,26]).

Under an additional *a-priori* bound assumption, a continuous dependence of the solution on the Cauchy data can be obtained. This is called conditional stability (e.g. [2,21]). We note that the conditional stability is closely related with the convergence of some regularization methods. For example, in [6] Cheng et al. provided a relationship between the convergence rate of the Tikhonov regularization method and conditional stability for an ill-posed operator equation. In [16] and [17], based on the conditional stability, the authors gave some convergence estimates for gradient-based methods and general linear regularization methods to treat with a linear ill-posed operator equation. In this paper, based on the conditional stabilities of a Cauchy problem, we give a general proof on the convergence estimates for three special methods for solving the Cauchy problem: the nonlocal boundary value problem method, the boundary Tikhonov regularization method and a generalized method. The first two methods have been investigated extensively in [9,23] where the authors presented convergence analysis based on the direct error estimates without using conditional stability. As we know, the convergence proof based on conditional stability for the Cauchy problem of Laplace equation is new issue and the generalized regularization method does not appear in references.

In [10], Eldén et al. gave an explicit and concrete stability result for problem (1.1), (1.3)–(1.4) with the homogenous Neumann condition at boundary $x = 0$ and $x = \pi$ in a square domain. By the method in [10] and a small modification, the stability estimate for a solution of problem (1.1)–(1.4) in a rectangular domain is also obtained which has a little difference from one in [10] and we show it in the following proposition.

Proposition 1.1. *Assume that the function u satisfies* (1.1)–(1.4) *and*

$$\int_0^\pi \varphi^2 dx \le \varepsilon, \qquad \|u\|^2_{L^2(D_a)} \le M,$$