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Finite Element Approximation of Semilinear Parabolic Optimal Control Problems

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Abstract. In this paper, the finite element approximation of a class of semilinear parabolic optimal control problems with pointwise control constraint is studied. We discretize the state and co-state variables by piecewise linear continuous functions, and the control variable is approximated by piecewise constant functions or piecewise linear discontinuous functions. Some *a priori* error estimates are derived for both the control and state approximations. The convergence orders are also obtained.

AMS subject classifications: 49J20, 65M60

Key words: Finite element approximation, semilinear parabolic optimal control, pointwise control constraint, *a priori* error estimates.

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1. Introduction

Optimal control problems have been widely studied and applied in science and engineering numerical simulation. The finite element method seems to be the most widely used numerical methods in computing optimal control problems. More recently, there have been extensively studies in the finite element approximation of the general optimal control problems, see, for example, [3–5,11–18] and the references cited therein. However, it is impossible to give even a very brief review here. Systematic introductions of the finite element method for PDEs and optimal control problems can be found in, for example, [1, 2, 7–10].

In this work, we focus our attention on the finite element approximation of the following semilinear parabolic optimal control problems:

$$\min_{u\in K}\left\{\int_0^T \big(g(y)+h(u)\big)\,dt\right\}, \tag{1.1}$$

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subject to the state equation

$$\begin{cases} y_t - \mathrm{div}(A\nabla y) + \phi(y) = f + Bu, & \text{in } \Omega \times (0,T], \\ y(x,t) = 0, & \text{on } \partial\Omega \times (0,T], \\ y(x,0) = y_0(x), & \text{in } \Omega, \end{cases} \tag{1.2}$$

where $g(\cdot)$ and $h(\cdot)$ are two given convex functionals, K denotes the admissible set of the control variable u, and B is a linear continuous operator. The details will be specified later on. Problems (1.1)-(1.2) appears, for example, in temperature control problems, see [6].

In this paper, we aim to derive a L^2-norm error estimates for both the control and state approximations in space variables. Either piecewise constant elements ($m = 0$) or piecewise linear discontinuous elements ($m = 1$) for the control approximation is adopted. It is proved that these approximations have convergence order $\mathcal{O}(h_U^{1+m/2} + h^2 + \Delta t)$, where h_U and h are the spatial mesh-sizes for the control and state, respectively, and Δt is the time increment.

The remainder of this paper is organized as follows. In Section 2, we shall briefly discuss the finite element approximation for the semilinear parabolic control problems. In Section 3 some *a priori* error estimates are derived for both the control and state approximations. The paper ends with results from some numerical experiments in Section 4.

Throughout this work, we employ the usual notion for Lebesgue and Sobolev spaces, see [1,2] for details. In addition, c or C denotes a generic positive constant independent of the discrete parameters.

2. Finite element approximation of optimal control problems

In this section, we study the finite element approximation of problems (1.1)–(1.2). To describe it, let Ω and Ω_U be bounded open convex polygons in $\mathbb{R}^n$ ($n \le 3$), with Lipschitz boundaries $\partial\Omega$ and $\partial\Omega_U$. Let $I = (0,T]$ be the time interval, and partition it by $T = N_T\Delta t, N_T \in \mathbb{Z}$, with $t_i = i\Delta t$ for $1 \le i \le N_T$. Let $f^i = f(x,t_i)$. We define, for $1 \le q < \infty$, the discrete time-dependent norms

$$\|f\|_{l^q(I;W^{m,p}(\Omega))} = \left(\sum_{i=1}^{N_T} \Delta t \|f^i\|_{m,p}^q \right)^{\frac{1}{q}},$$

and the standard modification for $q = \infty$. Let

$$l^q(I;W^{m,p}(\Omega)) := \left\{ f : \|f\|_{l^q(I;W^{m,p}(\Omega))} < \infty \right\}, \qquad 1 \le q \le \infty.$$

We shall take the state space $W = L^2(I;V)$ with $V = H_0^1(\Omega)$, the control space $X = L^2(I;U)$ with $U = L^2(\Omega_U)$, and $H = L^2(\Omega)$ to fix the idea. Let B be a linear continuous operator from X to $L^2(I;V')$, and K be a closed convex set in X. Let $g(\cdot)$ be a convex functional which is continuous differentiable on the observation space $H = L^2(\Omega)$, and