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An Iterative Multigrid Regularization Method for Toeplitz Discrete Ill-Posed Problems

Marco Donatelli*

Dipartimento di Fisica e Matematica, Università dell'Insubria, Via Valleggio, 11, Como 22100, Italia.

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Abstract. Iterative regularization multigrid methods have been successful applied to signal/image deblurring problems. When zero-Dirichlet boundary conditions are imposed the deblurring matrix has a Toeplitz structure and it is potentially full. A crucial task of a multilevel strategy is to preserve the Toeplitz structure at the coarse levels which can be exploited to obtain fast computations. The smoother has to be an iterative regularization method. The grid transfer operator should preserve the regularization property of the smoother. This paper improves the iterative multigrid method proposed in [11] introducing a wavelet soft-thresholding denoising post-smoother. Such post-smoother avoids the noise amplification that is the cause of the semi-convergence of iterative regularization methods and reduces ringing effects. The resulting iterative multigrid regularization method stabilizes the iterations so that and imprecise (over) estimate of the stopping iteration does not have a deleterious effect on the computed solution. Numerical examples of signal and image deblurring problems confirm the effectiveness of the proposed method.

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Key words: Multigrid methods, Toeplitz matrices, discrete ill-posed problems.

1. Introduction

Signal and image deblurring is an important task with many applications [18]. The blurring may be caused by object motion, calibration errors of the devices, or random fluctuations of the medium. We are concerning with restoration of blurred and noisy signals. The blurring process can be formulated in the form of Fredholm integral equations of the first kind. Let the function g represent the available observed blur- and noise-contaminated signal and let the function f represent the associated (unavailable) blur- and noise-free

*Corresponding author. *Email address:* `marco.donatelli@uninsubria.it` (M. Donatelli)

signal that we would like to recover. A first kind Fredholm integral equation for a one dimensional problem is as follows:

$$g(s) = \int_{\Omega} K(s,t)f(t)\mathrm{d}t, \qquad s \in \Omega, \tag{1.1}$$

where the point spread function (PSF) K is known. In our applications, K is smooth, the integral operator is compact and its inverse is unbounded if it exists. The task of solving (1.1) hence is an ill-posed problem [14]. In particular, we assume a spatially invariant PSF, that is, its effect depends only on the distance between s and t, thus, with a slight abuse of notation, we have

$$K(s,t) = K(s-t).$$

Discretization of (1.1) yields to a linear system of equations

$$A\mathbf{f} = \mathbf{g} = \mathbf{g}^{\text{blur}} + \mathbf{e}, \tag{1.2}$$

where A represents the blurring operator, $\mathbf{g}$ the available noise- and blur-contaminated signal, $\mathbf{g}^{\text{blur}}$ the blurred but noise free signal, and $\mathbf{e}$ the noise. The matrix A is a real Toeplitz matrix thanks to the space invariant property of the PSF. The ill-posedness of the continuous problem (1.1) implies that the matrix A is ill-conditioned and its singular values decay to zero without significant spectral gap, thus the linear system (1.2) is refereed as a discrete ill-posed problem [22]. This implies that straightforward solution of the linear system (1.2) does not provide a meaningful approximation of the desired signal because of the presence of the noise. A meaningful approximation of $\mathbf{f}$ can be determined by first replacing (1.2) with a nearby problem whose solution is less sensitive to perturbations in the data $\mathbf{g}$. This method is called regularization. Regularization methods include Tikhonov regularization or early termination of certain iterative methods [14]. Iterative methods provide an attractive alternative to Tikhonov regularization for large-scale problems [20]. When applied to ill-posed problems, many iterative methods exhibit a semiconvergence behavior. Specifically, the early iterations reconstruct information about the solution, while later iterations reconstruct information about the noise. The iteration number can be thought of as a discrete regularization parameter. A regularized solution is obtained by terminating the iterations after suitably few steps when the restoration error is minimized. Parameter selection methods such as discrepancy principle, GCV, and L-curve can be used to estimate the termination iteration [22]. The difficulty is that these techniques are not perfect and an imprecise estimate of the termination iteration can result in a solution whose relative error is significantly higher than the optimal, especially if the convergence is too fast and the restoration error curve is steep. Conjugate gradient type methods give reasonable results when applied to signal/image deblurring, but often they cut-off the high frequencies failing to recover the edges accurately.

Multigrid methods have already been considered to solve ill-posed problems [4, 5, 10, 21, 25, 26, 31]. They are usually applied to Tikhonov like regularization methods and no as iterative regularization methods. The first attempt in this direction was probably