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Multigrid Methods for Elliptic Optimal Control Problems with Pointwise State Constraints

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Abstract. An elliptic optimal control problem with constraints on the state variable is considered. The Lavrentiev-type regularization is used to treat the constraints on the state variable. To solve the problem numerically, the multigrid for optimization (MGOPT) technique and the collective smoothing multigrid (CSMG) are implemented. Numerical results are reported to illustrate and compare the efficiency of both multigrid strategies.

AMS subject classifications: 49K20, 65N06, 65N55

Key words: Elliptic optimal control problems, Lavrentiev regularization, multigrid methods, pointwise state constraints.

1. Introduction

Different numerical techniques solve elliptic optimal control problems efficiently. Multigrid is considered as one of the most efficient tools for solving elliptic type problems. As evidence, previous results show that multigrid solves optimal control problems with optimal computational complexity. See for example the application of multigrid to unconstrained optimization problems [13, 17], to optimal control problems [4, 5, 7, 12] and to inverse problems [18, 19]. The purpose of this paper is to formulate a fast numerical technique for solving state-constrained optimal control problems. These type of problems are very important in different applications of optimal control of partial differential equations. We focus on two representatives of multigrid methods for solving state-constrained optimal control problems: the multigrid for optimization (MGOPT) technique and the collective smoothing multigrid (CSMG). The CSMG scheme solves optimal control problems by solving the corresponding optimality system. This approach needs to customize the collective smoothing strategy for each individual problem. Nevertheless, an appropriate design of the CSMG components results in a robust algorithm with typical multigrid efficiency [6].

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On the other hand, the MGOPT method was first introduced in [13,17]. In this scheme the multigrid solution process represents the outer loop where the control function is considered as the unique dependent variable. The inner loop consists of a classical one-grid optimization scheme. We consider the application of these multigrid methods for solving state-constrained elliptic optimal control problems. This work is an extension of [22], which is the case of control-constrained elliptic optimal control problems. For the state-constrained case, there are several well-known techniques available. Take for example the Lavrentiev-type regularization and the Moreau-Yosida regularization, together with numerical solvers like the interior point methods and the active set strategies [1, 2, 10, 14–16, 20, 21]. For optimal control problems with state constraints, the corresponding Lagrange multipliers are in general not contained in a function space but only given as measures [3, 8, 16]. In order to overcome this difficulty, a Lavrentiev-type regularization for the solution of state-constrained optimal control problems is used. The Lagrange multipliers associated with the regularized state constraints can be assumed to be functions in L^2 [14, 16, 20]. This type of regularization procedure approximates the state constraints by mixed control-state constraints. The solution of the regularized problem converges to the solution of the original problem for regularization parameters tending to zero [14, 16, 21].

In the next sections, state-constrained optimal control problems are presented together with the discretization scheme and a detailed description of appropriate smoothing and optimization algorithms. In Section 4, the multigrid scheme is formulated. Numerical experiments follow to demonstrate the ability of multigrid in solving state-constrained optimal control problems and a section of conclusion completes this paper.

2. Constrained optimal control problems

In this section, we discuss state-constrained elliptic optimal control problems. The corresponding optimality system is presented and the multigrid solution procedure is given in the next section.

A state-constrained optimal control problem governed by a partial differential equation can be formulated as follows:

$$
\begin{aligned}
&\min_{u\in L^2(\Omega)} J(y,u) := \frac{1}{2}\|y-z\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)},\\
&-\Delta y + F(y) + u = f \quad \text{in } \Omega,\\
&y = 0 \quad \text{on } \partial\Omega,\\
&\underline{y} \le y \le \overline{y} \quad \text{on } \partial\Omega,
\end{aligned}
\tag{2.1}
$$

where $\nu > 0$ is the weight of the cost of the control, $z \in L^2(\Omega)$ is the target function, $f \in L^2(\Omega)$ and the function F is twice continuously differentiable and monotonically increasing [9,15]. The bounds $\underline{y}$ and $\overline{y}$ are fixed functions in $L^2(\Omega)$, where $\underline{y} \le \overline{y}$ almost everywhere in Ω. The existence and uniqueness of a solution to a state-constrained optimal control problem depend on the nonlinearity and on the given constraints. See for example [9, 15, 20].