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# Quasi-Optimal Convergence Rate of an AFEM for Quasi-Linear Problems of Monotone Type

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**Abstract.** We prove the quasi-optimal convergence of a standard adaptive finite element method (AFEM) for a class of nonlinear elliptic second-order equations of monotone type. The adaptive algorithm is based on residual-type a posteriori error estimators and Dörfler's strategy is assumed for marking. We first prove a contraction property for a suitable definition of total error, analogous to the one used by Diening and Kreuzer (2008) and equivalent to the total error defined by Cascón et. al. (2008). This contraction implies linear convergence of the discrete solutions to the exact solution in the usual $H^1$ Sobolev norm. Secondly, we use this contraction to derive the optimal complexity of the AFEM. The results are based on ideas from Diening and Kreuzer and extend the theory from Cascón et. al. to a class of nonlinear problems which stem from strongly monotone and Lipschitz operators.

**AMS subject classifications**: 35J62, 65N30, 65N12

**Key words**: quasilinear elliptic equations, adaptive finite element methods, optimality.

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## 1. Introduction

The main goal of this article is the study of convergence and optimality properties of an adaptive finite element method (AFEM) for quasi-linear elliptic partial differential equations over a polygonal/polyhedral domain $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) having the form

$$\begin{cases} Au := -\nabla \cdot \left[\alpha(\,\cdot\,, |\nabla u|^2)\nabla u\right] = f & \text{in}\,\Omega \\ \qquad\qquad\qquad\qquad\qquad\;\; u = 0 & \text{on}\,\partial\Omega, \end{cases} \tag{1.1}$$

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where $\alpha : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ is a bounded positive function whose precise properties will be stated in Section 2 below, and $f \in L^2(\Omega)$ is given. The assumptions on $\alpha$ guarantee that the nonlinear operator $A$ is Lipschitz and strongly monotone; see (2.6)–(2.7). This kind of problems arises in many practical situations; see Section 2.2 below.

AFEMs are an effective tool for making an efficient use of the computational resources, and for certain problems, it is even indispensable to their numerical resolvability. The ultimate goal of AFEMs is to equidistribute the error and the computational effort obtaining a sequence of meshes with optimal complexity. Adaptive methods are based on a posteriori error estimators, that are computable quantities depending on the discrete solution and data, and indicate a distribution of the error. A quite popular, natural adaptive version of classical finite element methods consists of the loop

$$\textsc{Solve} \to \textsc{Estimate} \to \textsc{Mark} \to \textsc{Refine}, \tag{1.2}$$

that is: solve for the finite element solution on the current grid, compute the a posteriori error estimator, mark with its help elements to be subdivided, and refine the current grid into a new, finer one.

A general result of convergence for linear problems has been obtained by Morin, Siebert and Veeser [16], where very general conditions on the linear problems and the adaptive methods that guarantee convergence are stated. Following these ideas a (plain) convergence result for elliptic eigenvalue problems has been proved in [8]. On the other hand, optimality of adaptive methods using *Dörfler's marking strategy* [7] for linear elliptic problems has been stated by Stevenson [22] and Cascón, Kreuzer, Nochetto and Siebert [2]. Linear convergence of an AFEM for elliptic eigenvalue problems has been proved in [13], and optimality results can be found in [5, 9]. For a summary of convergence and optimality results of AFEM we refer the reader to the survey [18] and the references therein. We restrict ourselves to those references strictly related to our work.

Well-posedness and a priori finite element error estimates for problem (1.1) have been stated in [4]. A posteriori error estimators for nonconforming approximations have been developed in [19]. Linear convergence of an AFEM for the $\varphi$-Laplacian problem in a context of Sobolev-Orlicz spaces has been established in [6]. Recently, the (plain) convergence of an adaptive *inexact* FEM for problem (1.1) has been proved in [10], where only a discrete *linear* system is solved before each adaptive refinement; albeit with stronger assumptions on $\alpha$.

In this article we consider a standard adaptive loop of the form (1.2) based on classical residual-type a posteriori error estimators, where the Galerkin discretization for problem (1.1) is considered. We use the Dörfler's strategy for marking and assume a minimal bisection refinement. The goal of this paper is to prove the optimal complexity of this AFEM by stating two main results. The first one establishes the linear convergence of the adaptive loop through a contraction property. More precisely, we will prove the following

**Theorem 1.1** (Contraction property). *Let $u$ be the weak solution of problem* (1.1) *and let $\{U_k\}_{k\in\mathbb{N}_0}$ be the sequence of discrete solutions computed through the adaptive algorithm de-*