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Efficient Simulation of Wave Propagation with Implicit Finite Difference Schemes

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Abstract. Finite difference method is an important methodology in the approximation of waves. In this paper, we will study two implicit finite difference schemes for the simulation of waves. They are the weighted alternating direction implicit (ADI) scheme and the locally one-dimensional (LOD) scheme. The approximation errors, stability conditions, and dispersion relations for both schemes are investigated. Our analysis shows that the LOD implicit scheme has less dispersion error than that of the ADI scheme. Moreover, the unconditional stability for both schemes with arbitrary spatial accuracy is established for the first time. In order to improve computational efficiency, numerical algorithms based on message passing interface (MPI) are implemented. Numerical examples of wave propagation in a three-layer model and a standard complex model are presented. Our analysis and comparisons show that both ADI and LOD schemes are able to efficiently and accurately simulate wave propagation in complex media.

AMS subject classifications: 35L05, 65M06, 765Y05

Key words: Acoustic wave equation, implicit schemes, ADI, LOD, stability condition, dispersion curve, MPI parallel computations.

1. Introduction

A basic and yet important problem in geophysical exploration is to determine the response to the excitation of an impulsive source. This step involves the numerical solution of the wave equation. In seismic exploration and imaging, modeling of wave mechanisms by the acoustic wave equation is accurate and widely used (Claerbout, 1985). There are four important techniques for the simulation of wave propagation: the finite element method (Ciarlet, 1978; Cohen, *et al.*, 2001), the discontinuous Galerkin method (Chung, *et al.*,

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2006,2009), the Fourier method (Fornberg, 1975,1990; Orszag 1980; Gazdag, 1981), and the finite difference method. In this paper, we will study the finite difference method.

The simulation of waves by the finite difference method was introduced as early as 1974 by Alford (Alford, 1974). Since then, many authors have made contributions to this direction, to list a few, Bayliss, 1986; Fornberg, 1990; Levander, 1988; Marfurt, 1984; Virieux, 1984, 1986; Sei, 1995; Minkoff, 2002. Among them, the staggered-grid finite difference method has become the most popular method for acoustic or elastic wave simulation. The staggered-grid finite difference method, which solves the velocity and stress simultaneously, is an explicit scheme proposed by Virieux in 1984 (Virieux, 1984). It has several advantages for seismic exploration modeling (Levander, 1988): (1) It is stable for all values of Poisson's ratio. Thus it is ideal for problems in which the materials concerned have high Poisson's ratio. (2) It has relatively small grid dispersion and grid anisotropy, and is relatively insensitive to Poisson's ratio. (3) It can easily incorporate with the free-surface boundary conditions. The dispersion relation and stability condition of the staggered-grid schemes are given by Sei (Sei, 1995). Fornberg compared the accuracy of high-order staggered-grid finite difference scheme with the pseudospectral method (Fornberg, 1988). In general, explicit finite difference schemes have good computational efficiency, however, they are only conditionally stable.

Contrary to explicit schemes, implicit schemes are not very popular in the simulation of wave propagation due to the fact that implicit schemes typically have lower computational efficiency. In particular, at each time step, an implicit scheme requires the solution of large linear systems. While these linear systems are diagonal or banded with small bandwidth for problems in one spatial dimension, they are neither diagonal nor narrow-banded for problems in higher spatial dimensions. Hence solving them requires significant amount of computational time. However, implicit schemes still attract some attention due to their unconditional stability. Using the splitting technique, which is commonly used for parabolic problems (Thomas 1995), we may split the two (or three) dimensional problem into several one dimensional problems. Therefore, we only need to invert banded linear systems with small bandwidth.

It is our main goal in this paper to investigate this approach for enhancing efficiency of implicit schemes. We will consider two implicit schemes, the alternating direction implicit (ADI) scheme and the locally one dimensional (LOD) scheme (Fairweather and Mitchell, 1965; Samarskii, 1964). Our aims are the investigation of the error bounds, stability conditions, and dispersion curves of ADI and LOD. Moreover, we will apply a spatial parallel scheme for the numerical computations with the message passing interface (MPI). Numerical tests for a three-layer model and the benchmark Marmousi model are performed. Our results demonstrate that the wave propagation phenomena are simulated accurately by the ADI and LOD schemes.

The paper is organized as follows. The error analysis and unconditional stability for ADI and LOD schemes with arbitrary spatial accuracy are established in Sections 2 and 3 respectively. In Section 4, dispersion analysis and curves both for ADI and LOD schemes are given. In Section 5, we present accuracy comparisons of ADI and LOD schemes. In Section 6, parallel computations based on MPI environment for a three-layer model and