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Convergence Analysis of a Block-by-Block Method for Fractional Differential Equations

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Abstract. The block-by-block method, proposed by Linz for a kind of Volterra integral equations with nonsingular kernels, and extended by Kumar and Agrawal to a class of initial value problems of fractional differential equations (FDEs) with Caputo derivatives, is an efficient and stable scheme. We analytically prove and numerically verify that this method is convergent with order at least 3 for any fractional order index $\alpha > 0$.

AMS subject classifications: 26A33, 65M12

Key words: Fractional differential equation, Caputo derivative, block-by-block method, convergence analysis.

1. Introduction

Fractional calculus [13, 14], almost as old as the familiar integer-order calculus, is now winning more and more scientific applications owing to its "memory" and "heredity" principle in a variety of areas, such as viscoelasticity [2], anomalous diffusion [3], control theory [15], finance [8, 16, 17] and hydrology [1, 18]. A recent panoramic view of the fractional calculus can be seen in [19].

Similarly to the integer-order differential equations, it is usually difficult to obtain the analytical solution for a fractional differential equation (FDE). So there has been a growing interest to develop numerical approaches in solving the FDEs. However, the theoretical studies of fractional numerical methods, including stability analysis and error estimation, are quite challenging due to the nonlocal property of fractional operators [5, 7, 12]. In this context, Diethelm *et al* [5, 7] took advantage of the fact that some kinds of FDEs can be formulated as Volterra integral equations of the second kind, then derived the fractional

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Adams-Bashforth-Moulton method from the classical case. Significantly, they gave convergence analysis, i.e., for any $\alpha > 0$ the described method is convergent with order at least one if the analytical solution $y(t)$ is twice continuously differentiable. In addition, Lin and Liu [12] developed a kind of linear multistep methods for fractional initial value problems based on Lubich's high-order approximations [10] to fractional derivatives and integrals. And they proved the consistence, convergence and stability of these methods. Nevertheless, the unavoidable shortcoming in these linear multistep methods is that one needs to spend much time in computing the starting weights.

In 2006, Kumar and Agrawal [9] also utilized the equivalent Volterra integral equation in [5] and extended the block-by-block method proposed by Linz [11] to some kinds of FDEs. Numerical examples have shown the efficiency and stability of this scheme, i.e., for a kind of FDEs the performance is better than that of Diethlm's Adams method [7]. However, it's a pity that the error estimate and convergence order analysis of this scheme was neglected. In the present paper, we will derive error estimate and precise convergence order of the block-by-block method under certain assumptions, and test the order via numerical experiments.

This paper is organized as follows. In Section 2, in order to facilitate the theoretical analysis, the block-by-block method is rewritten. We give in Section 3 some preparations and useful lemmas. The error estimate and convergence order analysis are given in Section 4. Numerical experiments are carried out in Section 5, which verify the theoretical results obtained in Section 4. Final section is the concluding remarks.

2. Block-by-block method

We consider the following nonlinear FDE

$$D_*^\alpha y(t) = f(t, y(t)), \quad 0 \le t \le T, \quad n-1 < \alpha \le n \tag{2.1}$$

subject to the initial conditions:

$$y^{(k)}(0) = c_k, \quad k = 0, 1, \cdots, n-1. \tag{2.2}$$

In (2.1), D_*^α denotes the Caputo derivative of order α, defined by

$$D_*^\alpha y(t) := \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{d^n y(\tau)}{d\tau^n} d\tau.$$

Assume that $\Omega := [0, T] \times [c_0 - \lambda, c_0 + \lambda]$ with some $\lambda > 0$ and $f(t, y) \in C(\Omega)$. Furthermore, let f fulfill a Lipschitz condition with respect to the second variable on Ω, namely

$$|f(t, y) - f(t, z)| \le L|y - z|$$

for some constant $L > 0$. According to [6], there exists a unique solution $y(t)$ on $[0, T]$ for the initial value problem (IVP) (2.1-2.2).