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REVIEW ARTICLE

A Review of Unified A Posteriori Finite Element Error Control

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Abstract. This paper aims at a general guideline to obtain a posteriori error estimates for the finite element error control in computational partial differential equations.
In the abstract setting of mixed formulations, a generalised formulation of the corresponding residuals is proposed which then allows for the unified estimation of the respective dual norms. Notably, this can be done with an approach which is applicable in the same way to conforming, nonconforming and mixed discretisations. Subsequently, the unified approach is applied to various model problems. In particular, we consider the Laplace, Stokes, Navier-Lamé, and the semi-discrete eddy current equations.

AMS subject classifications: 65N30, 65N15, 35J25

Key words: A posteriori, error analysis, finite element method, nonconforming finite element method, mixed finite element method, adaptive algorithm, Poisson equation, Lamé equations, Stokes equations, Maxwell equations, unified a posteriori error analysis, discontinuous Galerkin, residual estimator.

Contents

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1. Introduction

Numerical simulation in engineering and science involves all kinds of errors ranging from the modeling of the problem to round-off errors. We are concerned with problems that can be formulated as the linear equation

$$\mathscr{A}(p,u)=\ell$$

in function spaces Q and V with $\mathscr{A}(p,u)$ and ℓ in $(Q\times V)^*$. This paper is devoted to the control of the discretisation error that arises from the fact that the (unknown) exact solution $(p,u)\in Q\times V$ is approximated by a discrete solution (p_ℓ,u_ℓ) computed in a finite dimensional vector space $Q_\ell\times V_\ell$. The aim of a posteriori error control is the computation and justification of lower and upper error bounds for the unknown discretisation error $e:=(p,u)-(p_\ell,u_\ell)$. Apart from the standard case that the discrete spaces are subspaces of their infinite dimensional counterparts, we will also consider a violation of this inclusion in the case of non-conforming methods.

This paper is organized as follows. After some basic notations and definitions in this introductory section, a unifying formulation for different problem classes is introduced and applied to various examples in Section 2. Section 3 about the basic concepts of residual type error estimation is followed by a brief introduction to finite element spaces and interpolation operators in Section 4. The theoretical part is concluded by a compilation of crucial theorems in error estimation with proofs. In addition to the above statements, Sections 2 and 3 prepare applications discussed in detail in Sections 6 to 9.

Notation. In this paper, $a\lesssim b$ abbreviates $a\le Cb$ with some multiplicative mesh-size independent constant $C>0$ which only depends on the domain Ω and the shape (but not on the size) of finite element domains. Moreover, C is independent of crucial parameters of the partial differential equation (PDE) such as the Lamé parameter λ in the problem of linear elasticity below. Furthermore, $a\approx b$ abbreviates $a\lesssim b\lesssim a$.

Colon denotes the Euclidean scalar product of two matrices $A=(A_{jk}),B=(B_{jk})\in\mathbb{R}^{n\times n}$, that is, $A:B:=\sum_{j,k=1}^n A_{jk}B_{jk}$, the dyadic product of some vectors $a,b\in\mathbb{R}^n$ is denoted by $a\otimes b:=ab^T$ and the cross product of two vectors $a,b\in\mathbb{R}^3$ is written as $a\wedge b$. The space of symmetric matrices in $\mathbb{R}$ is defined by

$$\mathbb{R}^{n\times n}_{\mathrm{sym}}:=\{A\in\mathbb{R}^{n\times n}:A=A^T\}.$$

There are different definitions of the differential operator curl which we use in the doc-