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Recovery Type A Posteriori Error Estimates of Fully Discrete Finite Element Methods for General Convex Parabolic Optimal Control Problems

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Abstract. This paper is concerned with recovery type a posteriori error estimates of fully discrete finite element approximation for general convex parabolic optimal control problems with pointwise control constraints. The time discretization is based on the backward Euler method. The state and the adjoint state are approximated by piecewise linear functions and the control is approximated by piecewise constant functions. We derive the superconvergence properties of finite element solutions. By using the superconvergence results, we obtain recovery type a posteriori error estimates. Some numerical examples are presented to verify the theoretical results.

AMS subject classifications: 35B37, 49J20, 65N30

Key words: General convex optimal control problems, fully discrete finite element approximation, a posteriori error estimates, superconvergence, recovery operator.

1. Introduction

It is well known that finite element methods are undoubtedly the most widely used numerical method in computing optimal control problems. A systematic introduction of finite element methods for PDEs and optimal control problems can be found in [7, 15, 17–20, 23, 28, 32, 33]. The literature on a posteriori error estimation of finite element method is huge. Some internationally known works can be found in [1–4, 6]. Concerning finite element methods of elliptic optimal control problems, a posteriori error estimates of residual type were investigated in [26], a posteriori error estimates of recovery type were derived in [21].

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For parabolic optimal control problems, a priori error estimates of space-time finite element discretization were investigated in [29,30], a priori error estimates of finite element methods were established in [24], and residual type a posteriori error estimates of finite element methods were established in [27,34]. Recently, Fu and Rui considered a characteristic finite element approximation of control problems governed by transient advection-diffusion equations in [16].

Superconvergence properties of finite element methods for elliptic optimal control problems were studied in [10,11,31]. Yang and Chang showed the superconvergence properties for optimal control problem of bilinear type in [35]. The superconvergence of optimal control problems governed by Stokes equations were derived in [25]. Some superconvergence results of mixed finite element methods for elliptic optimal control problems can be found in [5,8,9,12,13,36]. Recently, we discussed the superconvergence of finite element methods for quadratic parabolic optimal control problems in [14].

The purpose of this work is to study the superconvergence and recovery type a posteriori error estimates of the fully discrete finite element approximation for general convex parabolic optimal control problems with control constraints.

We are interested in the following parabolic optimal control problem:

$$\begin{cases} \min\limits_{u(x,t)\in K} \left\{ \displaystyle\int_0^T \big(g(y(x,t)) + h(u(x,t))\big)\, dt \right\}, \\ y_t(x,t) - \operatorname{div}(A(x)\nabla y(x,t)) = f(x,t) + Bu(x,t), \quad x \in \Omega,\ t \in J, \\ y(x,t) = 0, \quad x \in \partial\Omega,\ t \in J, \\ y(x,0) = y_0(x), \qquad x \in \Omega, \end{cases} \tag{1.1}$$

where Ω be a bounded domain in $\mathbb{R}^2$ with a Lipschitz boundary $\partial\Omega$, $0 < T < +\infty$ and $J = [0,T]$. $g(\cdot)$ and $h(\cdot)$ are convex functionals on $L^2(\Omega)$. The coefficient $A(x) = (a_{ij}(x))_{2\times 2} \in (W^{1,\infty}(\Omega))^{2\times 2}$, such that for any $\xi \in \mathbb{R}^2$, $(A(x)\xi)\cdot\xi \geq c \mid \xi \mid^2$ with $c > 0$. Let B be a continuous linear operator from $L^2(\Omega)$ to $L^2(\Omega)$ and $f(x,t) \in C(J;L^2(\Omega))$. Moreover, we assume that $g(\cdot)$ is bounded below, $h(u) \to +\infty$ as $\|u\|_{L^2(\Omega)} \to \infty$ and K is a nonempty closed convex set in $L^2(J;L^2(\Omega))$, defined by

$$K = \Big\{ v(x,t) \in L^2(J;L^2(\Omega)) : a \leq v(x,t) \leq b, \quad a.e.\ (x,t) \in \Omega \times J \Big\},$$

where a and b are constants.

In this paper, we adopt the standard notation $W^{m,q}(\Omega)$ for Sobolev spaces on Ω with norm $\|\cdot\|_{W^{m,q}(\Omega)}$ and seminorm $|\cdot|_{W^{m,q}(\Omega)}$. We set $H_0^1(\Omega) \equiv \Big\{ v \in H^1(\Omega) : v|_{\partial\Omega} = 0 \Big\}$ and denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$. We denote by $L^s(J;W^{m,q}(\Omega))$ the Banach space of L^s integrable functions from J into $W^{m,q}(\Omega)$ with norm $\|v\|_{L^s(J;W^{m,q}(\Omega))} = (\int_0^T \|v\|^s_{W^{m,q}(\Omega)} dt)^{1/s}$ for $s \in [1,\infty)$ and the standard modification for $s = \infty$. We can define the space $H^l(J;W^{m,q}(\Omega))$. The details can be found in [24]. In addition, c or C denotes a generic positive constant independent of h and Δt.

The plan of the paper is as follows. In Section 2, we formulate the fully discrete finite element approximation for general convex parabolic optimal control problems. In Section