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Residual Based A Posteriori Error Estimates for Convex Optimal Control Problems Governed by Stokes-Darcy Equations

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Abstract. In this paper, we derive a posteriori error estimates for finite element approximations of the optimal control problems governed by the Stokes-Darcy system. We obtain a posteriori error estimators for both the state and the control based on the residual of the finite element approximation. It is proved that the a posteriori error estimate provided in this paper is both reliable and efficient.

AMS subject classifications: 65N15, 65N30

Key words: Optimal control, Stokes-Darcy equations, a posteriori error estimate.

1. Introduction

Flow control problems have received significant attention because of their many engineering applications. Extensive research has been carried out on various theoretical aspects of flow control problems, see, for example, [1–3] and the references therein, for existence results of optimal control, optimality conditions, regularity of optimal solutions and the existence of Lagrange multipliers.

It is obvious that efficient numerical methods are essential to successful applications of control problems. Nowadays, the finite element method is undoubtedly the most widely used numerical method in computing optimal control problems. There exists much literature on the finite element approximation for PDEs and various optimal control problems, see, for example, [4–11].

A posteriori error estimates are computable quantities in terms of the discrete solution or data, allow to measure the actual discrete errors without the knowledge of exact solutions. They are essential in designing algorithms with adaptive mesh refinement which

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equi-distribute the computational effort and optimize the approximation efficiency. Ever since the pioneering work of Babuška [12–14], the adaptive finite element method based on a posteriori error estimates has been extensively investigated. The literature in this area is huge. The work related on the a posteriori error estimates for partial differential equations is summarized in [15, 16]. In [17] two a posteriori error estimators for the mini-element discretization of the Stokes equations were presented.

Recently, the residual-based a posteriori error estimators for Stokes-Darcy coupled problems were presented in [18] and [19]. In [18], Bernardi-Raugel and Raviart-Thomas elements for the velocity and piecewise constants for the pressures were considered. In [19], we consider a posteriori error estimate for the Stokes-Darcy system with the Beavers-Joseph-Saffman-Jones interface boundary condition, where the approximation spaces are the Hood-Taylor element and the piecewise quadratic element for the Stokes and the Darcy regions, respectively.

Concerning the finite element approximation of the distributed optimal control problem governed by partial differential equations, residual based a posteriori error estimates are investigated by [20–23]. Especially, the optimal control problem governed by the Stokes equations was discussed in [3] and [24]. Moreover, we should also mention the dual weighted residual estimates pioneered by R. Becker and R. Rannacher (see, e.g., [25]). Recently, more work on this kind of problems can be found in [26, 28]. To our knowledge, there are still no theoretical results on a posteriori error estimates for the optimal control problem governed by the Stokes-Darcy system.

In this paper, we extend the result of [19] to the optimal control problems, we develop the a posteriori error analysis for the optimal control problem governed by the Stokes-Darcy system with the Beavers-Joseph-Saffman-Jones interface boundary condition. The approximation spaces for the state equations are the Hood-Taylor element and the piecewise quadratic element for the Stokes and the Darcy regions, respectively, while the control is approximated by piecewise constants space. We obtain the residual based a posteriori error estimators for both state and the control based on the residual of the finite element approximation. It is proved that the a posteriori error estimate provided in this paper is both reliable and efficient. Some techniques used in this paper can be found in, e.g., [3,9,15,16] and [19].

The rest of the paper is organized as follows: in Section 2, we shall construct a weak formulation and finite element approximation for the distributed optimal control problem governed by the Stokes-Darcy system. In Section 3, a posteriori error estimates in H^1-norm are derived for optimal control problems governed by the Stokes-Darcy system with the Beavers-Joseph-Saffman-Jones interface boundary condition. Reliability and efficiency are obtained in Subsections 3.1 and 3.2, respectively.

2. Finite element approximation of the control problems

Let Ω be an open bounded set in R^2 with piecewise Lipschitz boundary $\partial\Omega$. In this paper, we adopt the notation $W^{m,q}(\Omega)$ for Sobolev spaces on Ω with the standard norm $\|\cdot\|_{m,q,\Omega}$ and semi-norm $|\cdot|_{m,q,\Omega}$. For the case of $q=2$, we denote $H^m(\Omega)=W^{m,2}(\Omega)$ and