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Multi-Label Markov Random Fields as an Efficient and Effective Tool for Image Segmentation, Total Variations and Regularization

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Abstract. One of the classical optimization models for image segmentation is the well known Markov Random Fields (MRF) model. This model is a discrete optimization problem, which is shown here to formulate many continuous models used in image segmentation. In spite of the presence of MRF in the literature, the dominant perception has been that the model is not effective for image segmentation. We show here that the reason for the non-effectiveness is due to the lack of access to the optimal solution. Instead of solving optimally, heuristics have been engaged. Those heuristic methods cannot guarantee the quality of the solution nor the running time of the algorithm. Worse still, heuristics do not link directly the input functions and parameters to the output thus obscuring what would be ideal choices of parameters and functions which are to be selected by users in each particular application context.

We describe here how MRF can model and solve efficiently several known continuous models for image segmentation and describe briefly a very efficient polynomial time algorithm, which is provably fastest possible, to solve optimally the MRF problem. The MRF algorithm is enhanced here compared to the algorithm in Hochbaum (2001) by allowing the set of assigned labels to be any discrete set. Other enhancements include dynamic features that permit adjustments to the input parameters and solves optimally for these changes with minimal computation time. Several new theoretical results on the properties of the algorithm are proved here and are demonstrated for images in the context of medical and biological imaging. An interactive implementation tool for MRF is described, and its performance and flexibility in practice are demonstrated via computational experiments.

We conclude that many continuous models common in image segmentation have discrete analogs to various special cases of MRF and as such are solved optimally and efficiently, rather than with the use of continuous techniques, such as PDE methods, that restrict the type of functions used and furthermore, can only guarantee convergence to a local minimum.

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1. Introduction

Partitioning and grouping of similar objects plays a fundamental role in image segmentation and in clustering problems. In such problems the goals are to group together similar objects, or pixels in the case of image processing. Given an input image, the objective of image segmentation is to recognize the salient features in the image. Each feature set is grouped together in one segment represented by some uniform color area.

A noisy or corrupted image is characterized by lacking uniform color areas, which are assumed to characterize a true image. Rather, in such image there are adjacent pixels of different color areas. To achieve higher degree of uniform color areas, it is reasonable to assign a penalty to neighboring pixels that have different colors associated with them. On the other hand, the purpose of the segmentation is to represent the "true" image. For that purpose the given assignment of colors in the input image is considered to be the "priors" on the colors of the pixels, and as such, the best estimate available on their true labels. Therefore, any change in those priors is assigned a penalty for deviating from the priors.

The Markov Random Fields problem for image segmentation is to assign colors to the pixels so that the total penalty is minimized. The penalty consists of two terms. One is the *separation* penalty, or *smoothing* term, and the second is the *deviation* penalty, or *fidelity* term. For this reason we refer to this penalty minimization problem also as the *separation-deviation* problem. This problem has been extensively studied over the past two decades, see, e.g., [4, 7, 19, 20, 28, 31].

The input to the problem is a graph $G = (V, E)$, where in the case of image segmentation each pixel is represented as a node in V. Let $N(i)$ be the set of neighbors of node $i \in V$. For each $j \in N(i)$ the pair of nodes $\{i, j\}$ have an edge $[i, j] \in E$ connecting them. Each node $i \in V$ has a deviation function $G_i()$ associated with it, and each edge $[i, j] \in E$ has an associated separation function $F_{ij}()$. The problem formulation, described in full detail in Section 5 is,

$$\text{(MRF)} \quad \min \sum_{i \in V} G_i(x_i) + \sum_{i \in V} \sum_{j \in N(i)} F_{ij}(x_i - x_j)$$
$$\text{subject to} \quad x_i \in X_i, \quad \forall i \in V.$$

The sets X_i consists of a collection of discrete labels, $\{a_1, a_2, \cdots, \mathbf{a}_n\}$, that can be assumed by variables x_i in a feasible solution. That is, MRF is a *multi-label* assignment. It is noted that the concept of "colors" associated with pixels can be replaced by any other scalar characterization of pixels or voxels, such as texture. We refer here to colors as a representation of such characterizations.

The complexity of MRF depends on the form of the penalty functions. This complexity of MRF was fully resolved and classified according to the properties of the penalty functions in [26] for sets of consecutive integers $X_i = \{\ell_i, \ell_{i+1}, \cdots, u_i\}$. For convex penalty functions MRF is polynomial time solvable, and for non-convex the problem is NP-hard. The cases when the deviation penalty functions are convex and the separation penalty functions are (bi-)linear (defined below) was shown by Hochbaum [26] to be solvable in polynomial