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Augmented Lagrangian Methods for *p*-Harmonic Flows with the Generalized Penalization Terms and Application to Image Processing

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Abstract. In this paper, we propose a generalized penalization technique and a convex constraint minimization approach for the p-harmonic flow problem following the ideas in [Kang & March, IEEE T. Image Process., 16 (2007), 2251–2261]. We use fast algorithms to solve the subproblems, such as the dual projection methods, primal-dual methods and augmented Lagrangian methods. With a special penalization term, some special algorithms are presented. Numerical experiments are given to demonstrate the performance of the proposed methods. We successfully show that our algorithms are effective and efficient due to two reasons: the solver for subproblem is fast in essence and there is no need to solve the subproblem accurately (even 2 inner iterations of the subproblem are enough). It is also observed that better PSNR values are produced using the new algorithms.

AMS subject classifications: 68U10, 65N21, 74S20

Key words: p-harmonic flows, denoising, generalized penalization terms, saddle-point problem, image processing, augmented Lagrangian methods.

1. Introduction

First we present the p-harmonic flow problem in [17, 28] as

$$\min_{\boldsymbol{U}\in \boldsymbol{W}^{1,p}(\Omega,S^{N-1})} \boldsymbol{E}(\boldsymbol{U}) = \int_\Omega |\nabla \boldsymbol{U}(\boldsymbol{x})|_F^p \mathrm{dx}, \tag{1.1}$$

where $1 \le p < \infty$. Some notations in (1.1) are defined as follows:

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- Ω: an open subset of $\mathscr{R}^M$.
- ∇: differential operator, i.e.,

$$\nabla \boldsymbol{U} = \Big(\frac{\partial U_i}{\partial x_j}\Big)_{N\times M}, \quad \nabla U_i = \Big(\frac{\partial U_i}{\partial x_1}, \cdots, \frac{\partial U_i}{\partial x_M}\Big), \quad \forall \boldsymbol{U} = (U_1, \cdots, U_N)^T \in \mathscr{R}^N.$$

- $|\cdot|$: Euclidean norm, and $|\cdot|_F$: Frobenius norm, i.e.,

$$|\boldsymbol{B}|_F = \sqrt{\sum_{i,j} B_{i,j}^2}, \quad \forall \boldsymbol{B} = (B_{i,j})_{N\times M}.$$

- $\boldsymbol{W}^{1,p}(\Omega, S^{N-1}) := \boldsymbol{W}^{1,p}(\Omega, \mathscr{R}^N) \cap S^{N-1}$,

$$S^{N-1} := \big\{\boldsymbol{U} \in \mathscr{R}^N : |\boldsymbol{U}| = 1, \ a.e.\big\}, \quad M \geq 1, \quad N \geq 2.$$

- $(\cdot)^T$ denotes the transpose of the matrix.

The minimization of (1.1) is associated with the Dirichlet boundary condition: $\boldsymbol{U}|_\Omega = \boldsymbol{n}_0 \in S^{N-1}$ or Neumann boundary condition: $\partial \boldsymbol{U}/\partial \boldsymbol{n} = 0$ where $\boldsymbol{n}$ is the exterior unit normal to $\partial\Omega$.

The difficulties of solving (1.1) lie in three aspects, i.e., the non-convexity due to constraints of S^{N-1}, the non-regularity and non-uniqueness. Several kinds of approaches are used to solve (1.1) in literature. The authors [14,15] dealt with the Euler-Lagrange equations for problem (1.1) using the iteration which updated the solution by normalizing $\boldsymbol{U} = \boldsymbol{V}/|\boldsymbol{V}|$. Analysis on the similar algorithms were done in [3–5] and constraints preserving finite element methods were proposed in [6,7]. In [19], the authors adopted the saddle-point approach and established the proper finite element discretization in the case of two dimensional space. The second kind of approach was proposed by adding a penalization to eliminate the non-convex constraint of S^{N-1} [8,9,24]. Such technique is also adopted to solve the Ginzburg-Landau functional, i.e.,

$$E_\epsilon(\boldsymbol{U}) := E(\boldsymbol{U}) + \frac{1}{\epsilon}\int_\Omega \Big(|\boldsymbol{U}^2| - 1\Big)^2 \,\mathrm{dx}. \tag{1.2}$$

The third kind of approach is to reformulate (1.1) to become a constrained optimization problem as follows

$$\min_{\boldsymbol{U}\in \boldsymbol{W}^{1,p}(\Omega,\mathscr{R}^N)} E(\boldsymbol{V}), \ \ s.t. \ \ \boldsymbol{V} = \frac{\boldsymbol{U}}{|\boldsymbol{U}|}. \tag{1.3}$$

Such constraint was used to preserve gradient descent for solving (1.3) in [10,28]. Further improvements based (1.3) were done in [17,29] in which the authors proposed an innovative curvilinear search method with the global convergence property as long as satisfying Armijo-Wolfe conditions.

In this paper, by combining the second and the third approach via the relaxation and penalization, a general model is established with penalization terms following the idea in [20]. We derive the saddle-point problem for (1.1) based on the augmented Lagrangian