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A New Discontinuous Galerkin Method for Parabolic Equations with Discontinuous Coefficient

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Abstract. In this paper, a new discontinuous Galerkin method is developed for the parabolic equation with jump coefficients satisfying the continuous flow condition. Theoretical analysis shows that this method is L^2 stable. When the finite element space consists of interpolative polynomials of degrees k, the convergent rate of the semi-discrete discontinuous Galerkin scheme has an order of $\mathcal{O}(h^k)$. Numerical examples for both 1-dimensional and 2-dimensional problems demonstrate the validity of the new method.

AMS subject classifications: 65M60, 35K05

Key words: Parabolic equation, discontinuous coefficient, discontinuous Galerkin method, error estimate, stability analysis.

1. Introduction

The parabolic equations with discontinuous coefficients play a great role in many physical applications. For example, in the numerical simulation of radiation hydrodynamics, since energy is usually transported in a variety of media, the conductivity coefficients are discontinuous on the media interface. Sometimes the conductivity coefficients are with several quantity differences. Therefore, it is of great theoretical significance and practical value to study the numerical methods with high order accuracy [1]. There have been some works for solving parabolic problems with discontinuous coefficients by finite difference methods, finite volume methods, and finite element methods. Samarskii [2] studied

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the classical θ-scheme. Shashkov [3] developed support-operators method to solve diffusion equations with rough coefficients. Zhu et al. [4] presented explicit/implicit schemes. Sinha et al. [5] studied the error estimates of finite element method. Huang and Li [6] gave the immersed methods combined with finite difference method and Ewing et al. [7] and Li et al. [8] combined immersed methods with finite element approximations to obtain the numerical solution of the interface problem. The Discontinuous Galerkin (DG) method was first introduced by Reed and Hill [9] for solving neutron transport problems. A major development of the DG method was carried out by Cockburn and Shu [10–13] for solving hyperbolic conservation laws. It now becomes an active research area for solving hyperbolic, elliptic and parabolic equations. The DG method uses a completely discontinuous piecewise polynomial as the solution and test function space, and it has the good properties: local conservation on each element, suitability for hp-adaptive implementation; easily treating rough coefficient problems and effectively capturing discontinuities. For time-dependent convection diffusion problems, DG methods provide substantial computational advantages if explicit time integrations are used. Motivated by the successful realization in solving hyperbolic equations, the DG method is applied to solve equations with high order derivatives and developed as the local discontinuous Galerkin (LDG) method [14], the DG method based on dGRP flux (diffusive generalized Riemann problem) [15,16], and direct discontinuous Galerkin (DDG) method [17, 18]. In recent years, the DG methods have been applied to solve the elliptic equations and advection-diffusion equations with discontinuous coefficients. Ern et al. [19–21] developed a (symmetric) weighted interior penalty (WIP) method which replaces the arithmetic mean with suitably weighted averages where the weights depend on the coefficients of the problem. Cai et al. [22] proposed three different weight averages and established a priori and a posteriori error estimates.

The present paper based on the continuous flow condition constructs a new discontinuous Galerkin method which satisfies the consistent numerical flux. The new DG method needs not introduce auxiliary variables and has high efficiency compared with the LDG method. In meantime, the new DG method can ensure high accuracy and stability and sharply capture the contacts with discontinuous derivatives.

The paper is organized as follows. In Section 2, we construct the new DG scheme for one-dimensional heat conduction equation with jump coefficient and prove the L^2 stability and error estimate. In Section 3, this DG method is extended to two-dimensional heat conduction equation. The stability and convergence analysis are studied. Numerical examples are presented in Section 4 to illustrate the efficiency and accuracy of the new method. Some conclusions are given in the last section.

2. DG scheme for 1D problem

Consider the 1D heat conduction equation:

$$\rho c \frac{\partial U}{\partial t} - \frac{\partial}{\partial x}\Big(\kappa(x)\frac{\partial U}{\partial x}\Big) = f(x,t), \quad x_L < x < x_R, \quad t > 0 \tag{2.1}$$