Numer. Math. Theor. Meth. Appl. Vol. **6**, No. 2, pp. 364-383
doi: 10.4208/nmtma.2013.1130nm May 2013

# A Numerical Study of Blowup in the Harmonic Map Heat Flow Using the MMPDE Moving Mesh Method

Ronald D. Haynes[1,*], Weizhang Huang[2] and Paul A. Zegeling[3]

[1] *Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, NL A1C 5S7, Canada.*
[2] *Department of Mathematics, University of Kansas, Lawrence, KS 66045, USA.*
[3] *Department of Mathematics, Utrecht University, Utrecht, The Netherlands.*

Received October 30, 2011; Accepted (in revised version) 18 April 2012

Available online 12 April 2013

**Abstract.** The numerical solution of the harmonic heat map flow problems with blowup in finite or infinite time is considered using an adaptive moving mesh method. A properly chosen monitor function is derived so that the moving mesh method can be used to simulate blowup and produce accurate blowup profiles which agree with formal asymptotic analysis. Moreover, the moving mesh method has finite time blowup when the underlying continuous problem does. In situations where the continuous problem has infinite time blowup, the moving mesh method exhibits finite time blowup with a blowup time tending to infinity as the number of mesh points increases. The inadequacy of a uniform mesh solution is clearly demonstrated.

**AMS subject classifications**: 35K91, 65M50, 65M06

**Key words**: Heat flow, harmonic map, blowup, moving mesh method, finite difference.

## 1. Introduction

We are concerned with the numerical solution of the harmonic map heat flow into the unit sphere. In particular, we consider the initial-boundary value problem (IBVP)

$$\theta_t = \theta_{rr} + \frac{1}{r}\theta_r - \frac{\sin(2\theta)}{2\,r^2}, \quad 0 < r < 1 \tag{1.1}$$

$$\theta(0,t) = 0, \quad \theta(1,t) = \theta_1, \tag{1.2}$$

$$\theta(r,0) = \theta^0(r), \quad 0 < r < 1, \tag{1.3}$$

where $\theta^0 \in C[0,1]$ is a given function satisfying $\theta^0(0) = 0$ and $\theta^0(1) = \theta_1$. It is known [12, 17, 18] that the solution of the IBVP (a) exists for all time if $|\theta^0(r)| < \pi$ for all

*Corresponding author. *Email addresses:* `rhaynes@mun.ca` (R. D. Haynes), `huang@math.ku.edu` (W. Huang), `P.A.Zegeling@uu.nl` (P. A. Zegeling)

 

$r \in [0,1]$, (b) blows up in a finite time and at infinity if $|\theta_1| > \pi$ and $|\theta_1| = \pi$, respectively, and (c) may blow up in a finite time if $|\theta_1| < \pi$ but $|\theta^0(r)|$ rises above $\pi$ for some $r \in (0,1)$. Here, a solution is said to blow up in a finite time or at infinity (denoted by $T \in (0,+\infty]$) if

$$\limsup_{t\uparrow T} \|\theta_r(\cdot,t)\|_\infty = +\infty. \tag{1.4}$$

This is different from the widely studied blowup phenomenon associated with semilinear parabolic equations where the solution itself, instead of the spatial derivative of the solution, becomes unbounded in a finite time (e.g., see Friedman and Mcleod [25]). Moreover, via formal asymptotical analysis van den Berg et al. [35] show that for $\theta_1 > \pi$, the blowup behavior of IBVP (1.1)-(1.3) is given by

$$\lim_{t\uparrow T} \theta\left(\frac{\mu\kappa(T-t)}{|\ln(T-t)|^2}, t\right) = 2\ \arctan(\mu), \quad \text{for all fixed } \mu > 0, \tag{1.5}$$

where $\kappa > 0$ is a constant and $\mu$ is called the kernel coordinate in the literature.

PDE (1.1) is a special case of the harmonic map heat flow

$$u_t = \Delta u + |\nabla u|^2 u, \tag{1.6}$$

where $u(\cdot,t) : D^2 \to S^2$, $D^2$ is the unit disk in two dimensions and $S^2$ is the unit sphere in three dimensions. Indeed, it is easy to verify that (1.6) reduces to (1.1) for the radially symmetric solution

$$u(r,\phi,t) = \begin{pmatrix} \cos(\phi)\sin(\theta(r,t)) \\ \sin(\phi)\sin(\theta(r,t)) \\ \sin(\theta(r,t)) \end{pmatrix}, \tag{1.7}$$

where $(r,\phi)$ are the polar coordinates for $D^2$ and $\theta(r,t)$ satisfies (1.1). PDE (1.6) is the gradient flow associated with the energy $\mathscr{E} = \frac{1}{2}\int_{D^2} |\nabla u|^2 dV$ and its steady solution is a harmonic map from $D^2$ to $S^2$. It is known [16, 32] that IBVP (1.1)-(1.3) admits a classical solution for a sufficiently smooth initial solution with small energy and a global weak solution for a sufficiently smooth initial solution with finite energy. Such a weak solution is unique if the energy is non-increasing along the flow [24] and is smooth except for at most finitely many singular space-time points where non-constant harmonic maps "separate" and a downward jump in the energy and blowup in the spatial derivative of the solution occur. Finite time blowup in the harmonic map heat flow has been a topic of extensive research; e.g., see [12, 18, 20, 21, 26, 35].

PDEs (1.1) and (1.6) also arise in several other applications. For example, in the study of the evolution of the director field in nematic liquid crystals (LCs) [36], $u$ in (1.6) models the mean orientation of the long axis of the molecules comprising the LC, and (1.1) is obtained by considering LCs in a cylindrical tube. PDE (1.6) appears in the study of soft ferromagnets mechanically at rest [13, 22] where $u$ represents the magnetization of the deformable body. The harmonic map heat flow is also used in the study of color image enhancement or denoising; e.g., see [33, 38].