Numer. Math. Theor. Meth. Appl. Vol. 6, No. 2, pp. 408-423
doi: 10.4208/nmtma.2013.1127nm May 2013

Superconvergence and Asymptotic Expansions for Bilinear Finite Volume Element Approximations

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Received 2 October 2011; Accepted (in revised version) 13 March 2012

Available online 12 April 2013

Abstract. Aiming at the isoparametric bilinear finite volume element scheme, we initially derive an asymptotic expansion and a high accuracy combination formula of the derivatives in the sense of pointwise by employing the energy-embedded method on uniform grids. Furthermore, we prove that the approximate derivatives are convergent of order two. Finally, numerical examples verify the theoretical results.

AMS subject classifications: 65M10, 65M08, 41A60

Key words: Isoparametric bilinear finite volume element scheme, asymptotic expansion, high accuracy combination formula, superconvergence.

1. Introduction

Finite volume (FV) method has been one of the most commonly used numerical methods for solving partial differential equations due to its many attractive properties, such as preserving local conservation of certain physical quantities (mass, energy) and so on. The finite volume element (FVE) method is one important member of FV method. In 1982, Li and Zhu presented a generalized difference scheme [1], and proved the error estimate in H^1 norm on quadrilateral grids. The trial and test spaces are, respectively, chosen as bilinear finite element space and piecewise constant space. It is so-called the isoparametric bilinear finite volume element scheme. In 1993, Schmidt and Kiel constructed two types of box (diagonal box, center box) schemes [2], and obtained the saturated convergent order in H^1 norm and the superconvergent result on parallelogram grids based on the analysis of the eigenvalue problem for any partition element. Later, Porsching and Chou proposed a

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"Covolume Method" [3,4] which is actually a FVE method and widely applied in computational fluid dynamic problems. Simultaneously, some symmetric FVE schemes [5,6], high order FVE schemes [7–9] and new FVE schemes for three dimensional problems [10,11] were presented by some researches.

For the isoparametric bilinear finite volume element scheme, its optimal L^2 error estimate [12] is got more behind that on H^1 estimate. Recently, Li and Lv proved the optimal L^2 error results [13–15] for this scheme. Although the superconvergence for the finite element methods is abundantly studied [16–18], there are only some researches on superconvergence about the isoparametric bilinear finite volume element [1,15,19,20]. Furthermore, they are almost in the sense of average instead of pointwise. It urges us to study the superconvergence in the sense of pointwise.

In present paper, the innovative idea of our work is that we derive an asymptotic expansion for the isoparametric bilinear finite volume element solution. The derivation includes the achievement of the integral formula for the bilinear functional $A(u-u_I,v)$, the introduction of a proper auxiliary variational problem, and the employment of the discrete Green function and the energy-embedded method. Furthermore, we derive a high accuracy combination formula of the derivatives in the sense of pointwise on uniform grids for the first time, and prove that the approximate derivatives are convergent of order two. Numerical examples confirm the theoretical results.

The remainder of this paper is organized as follows. In Section 2, we introduce the isoparametric bilinear finite volume element scheme and some convergent results. In Section 3, we derive the asymptotic expansion for our finite volume element solution. In Section 4, we present a high accuracy combination formula of the approximate derivatives in the sense of pointwise on uniform grids and the corresponding superconvergence. Finally, we display numerical experiments to support our conclusions.

2. The isoparametric bilinear finite volume element scheme

We consider the following model problem

$$\begin{cases} -\nabla\cdot(\kappa\nabla u)=f, & \mathbf{x}\in\Omega, \\ u=0, & \mathbf{x}\in\partial\Omega, \end{cases} \tag{2.1}$$

where $\Omega\subset\mathbb{R}^2$ is a convex polygonal domain with boundary $\partial\Omega$, $f(\mathbf{x})\in L^2(\Omega)$ and $\kappa(\mathbf{x})\in C^1(\Omega)$ satisfies

$$\kappa(\mathbf{x})\geq\kappa_0,$$

and κ_0 is a positive constant.

Let $\Omega_h=\{E_i,\ 1\leq i\leq M\}$ be the quadrilateral partition of Ω (see Fig. 1(a)), and $\mathscr{D}=\{P_i=(x_i^1,x_i^2),\ 1\leq i\leq N\}$ be the set of partition nodes in Ω_h, where M and N are, respectively, the numbers of elements and nodes. Denote $\Omega_h^*=\{b_{P_i},1\leq i\leq N\}$ as the dual partition of Ω_h, where b_{P_i} is the dual element (also called control volume) about node P_i (see Fig. 1(b)). In this paper, we always assume that Ω_h and Ω_h^* are all quasi-uniform, i.e.,

$$C_1h^2\leq S_E\leq C_2h^2,\quad E\in\Omega_h\quad\text{and}\quad C_1h^2\leq S_{P_i}\leq C_2h^2,\quad b_{P_i}\in\Omega_h^*,$$