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Model Adaptation Enriched with an Anisotropic Mesh Spacing for Nonlinear Equations: Application to Environmental and CFD Problems

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Abstract. Goal of this paper is to suitably combine a model with an anisotropic mesh adaptation for the numerical simulation of nonlinear advection-diffusion-reaction systems and incompressible flows in ecological and environmental applications. Using the reduced-basis method terminology, the proposed approach leads to a noticeable computational saving of the online phase with respect to the resolution of the reference model on nonadapted grids. The search of a suitable adapted model/mesh pair is to be meant, instead, in an offline fashion.

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1. Introduction and motivations

Many physical phenomena are characterized by the coexistence of different scales in space and time. The most complex phenomena, however, often take place only over small parts of the whole spatial configuration. For example, in some fluid dynamics applications, the peculiar geometry of the configuration triggers some complex flow features: water in a backward-facing step channel shows complex patterns only past the step, where recirculations on small scales and detachment of the flow occur; blood in an artery with an aneurysm exhibits intricated recirculation patterns in the aneurysmal sac. It is clear that a monolithic approach to these intrinsically complex problems is, in general, prohibitive from a computational viewpoint, whereas a more tailor-made approach seems more feasible. This justifies the increasing interest in *reduced-order modeling* techniques,

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such as, reduced-basis method [21], proper generalized decomposition [8], model reduction [3, 7, 31], compressed sensing [6], etc.

The approach that we propose to contain the computational cost relies on a combination of model with mesh adaptation. This was investigated preliminarly in [11]. In particular, if we are given a reduced model as well as an adapted grid, both capturing the essential and complex features of the problem at hand, we can show that an *online* computation based on this adapted model/mesh leads to an appreciable computational saving over the employment of the monolithic model on a uniform, sufficiently fine, mesh. Of course, the *offline* phase required to build up the "database" of adapted model/mesh, is a costly overhead which is, for this reason, confined to a preliminary step.

Concerning the model adaptation, the idea is to devise an adapted model which is derived from the monolithic model by dropping the terms which are more computationally expensive [3, 4, 30]: to fix ideas, in the backward-facing step configuration, we expect that the nonlinear term in the momentum equation of the Navier-Stokes system can be neglected in some parts of the domain. Which actual parts, however, cannot be determined a priori; only an a posteriori adaptation method can predict where the nonlinear term can be actually dropped.

As far as the mesh adaptation is concerned, we employ anisotropically adapted grids, that is, where both the size, shape and orientation of the triangles are adjusted to mate the strong directional features as well as the small-scale patterns exhibited by the flow at hand. The computational benefits of anisotropic mesh adaptation over isotropic adaptativity are already well established in the literature [12, 13, 15, 16, 18, 22–24, 32].

Both model and mesh adaptations are driven by suitable a posteriori error estimators. In particular, we exploit the potentiality of a goal-oriented approach to control in a straightforward way quantities which are of some importance in diverse physical contexts [2, 17, 28].

The outline of the paper is the following. In Section 2, we focus on the model adaptation. Section 3 deals with mesh adaptation: first, the anisotropic setting is introduced, and then the goal-oriented a posteriori analysis is provided. In Section 4, model and mesh adaptivities are merged. In the last section, the proposed approach is applied to the Navier-Stokes equations for incompressible flows.

2. A goal-oriented a posteriori model analysis for nonlinear problems

The standard mathematical approach to a goal-oriented analysis in a nonlinear framework is based on a reformulation of the problem at hand as a constrained minimization problem, hinging on a suitable Lagrangian functional [2, 17]. Critical issues to be tackled in the definition of the Lagrangian are the treatment of nonhomogeneous Dirichlet boundary data as well as the inclusion of possible stabilization terms in the discrete variational formulation. In particular, concerning the first issue, we resort to a penalty method (see, e.g., [1, 9]).

Let V and W be two real Hilbert function spaces associated with the computational domain $\Omega \subset \mathbb{R}^2$, and consider the general weak formulation of the differential problem at