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The Global Behavior of Finite Difference-Spatial Spectral Collocation Methods for a Partial Integro-differential Equation with a Weakly Singular Kernel

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Abstract. The z-transform is introduced to analyze a full discretization method for a partial integro-differential equation (PIDE) with a weakly singular kernel. In this method, spectral collocation is used for the spatial discretization, and, for the time stepping, the finite difference method combined with the convolution quadrature rule is considered. The global stability and convergence properties of complete discretization are derived and numerical experiments are reported.

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1. Introduction

We consider initial-boundary value problems of the form

$$u_t(\mathbf{x},t)=\int_0^t \beta(t-s)\Delta u(\mathbf{x},s)ds+f(\mathbf{x},t), \quad (\mathbf{x},t)\in\Omega\times(0,T], \tag{1.1}$$

where $\beta(t)=t^{-1/2}/\Gamma(1/2)$, which has a weak singularity at $t=0$ and $\Omega\equiv(-1,1)^2$, subject to the boundary condition

$$u(\mathbf{x},t)=0 \quad \text{on} \quad \partial\Omega, \quad t>0, \tag{1.2}$$

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and the initial condition

$$u(\mathbf{x},0)=u_0 \quad \text{in } \Omega. \tag{1.3}$$

Here $u_t(\mathbf{x},t)=\partial u(\mathbf{x},t)/\partial t$, Δ is the two-dimensional Laplacian operator, $\partial\Omega$ is the boundary of the unit square Ω and β is a real-valued and positive-definite kernel, i.e., $\beta \in L^{1,loc}(0,+\infty)$ and satisfies

$$\int_0^T\int_0^t \beta(t-s)\varphi(s)ds\varphi(t)dt \geq 0, \quad \forall T>0, \quad \varphi \in C([0,T]). \tag{1.4}$$

Equations of the form (1.1) arise in problems concerned with heat conduction in materials with memory, population dynamics, viscoelasticity and theory of nuclear reactors (see Mustapha [19–21] and reference therein). The numerical solution of problems of the type (1.1) was studied extensively in the literature. See, for instance, Mclean and Thomée [17], Mclean et al. [18] and Pani et al. [23, 24] for the positive-type kernels, Chen [3], Sanz-Serna [25], López-Marcos [14], Lubich et al. [16], Mclean and Mustapha [19], Lin and Xu [11, 12], Tang [28] for weakly singular kernels, Da [31, 32] for completely monotonic kernels and Da [33] for log-convex kernels.

As we know, spectral methods have become increasingly popular and been widely used in spatial discretization of PDEs owing to its high order of accuracy (cf. [1, 2, 4–7, 26, 29]). Some work has been done along this line and we particularly point out that Kim and Choi [9] proposed and analyzed a spectral collocation method for the PIDEs with a weakly singular kernel, the spatial discretization is based on the pseudo-spectral method and the temporal discretization by finite difference methods was considered. Lin and Xu [11] proposed a finite difference scheme in time and Legendre spectral method in space for fractional diffusion-wave equation. Meanwhile, Li and Xu [10] proposed a spectral method in both temporal and spatial discretizations for this equation. In those papers [9–11], the error bounds of discretization in time are valid only on finite time intervals and pointwise. From a practical point of view, it is more interesting and challenging to develop and analyze high-order methods for PIDEs in a long time period.

This paper, motivated by [30], is devoted to approximate the problems (1.1)-(1.3) using spectral collocation in each spacial direction for the spatial discretization. Then the resulting systems of integro-differential equations in the time variable are discretized using backward Euler method, combined with the convolution quadrature rule, by employing a different approach involving the z-transform with respect to time sequence, we derive the global stability properties and associated error estimates for large T. It should be noted that the z-transform with respect to time sequence was employed by Sanz-Serna [25]. Our result is related to but different from [25].

The outline of this paper is as follows. In the next section, we first introduce the Sobolev spaces on a square and then define several projection operators from Sobolev spaces onto the space of polynomials with degree less than an integer N. In Section 3 we introduce the z-transform of a sequence $\{f_n\}_0^\infty$ and collect some of its properties. In Section 4, we establish stability and convergence of the full discrete scheme for (1.1). Numerical results in Section 5 validate the theoretical prediction in Section 4.