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# A Fourier Companion Matrix (Multiplication Matrix) with Real-Valued Elements: Finding the Roots of a Trigonometric Polynomial by Matrix Eigensolving

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**Abstract.** We show that the zeros of a trigonometric polynomial of degree $N$ with the usual $(2N+1)$ terms can be calculated by computing the eigenvalues of a matrix of dimension $2N$ with real-valued elements $M_{jk}$. This matrix $\vec{\vec{M}}$ is a multiplication matrix in the sense that, after first defining a vector $\vec{\phi}$ whose elements are the first $2N$ basis functions, $\vec{\vec{M}}\vec{\phi} = 2\cos(t)\vec{\phi}$. This relationship is the eigenproblem; the zeros $t_k$ are the arccosine function of $\lambda_k/2$ where the $\lambda_k$ are the eigenvalues of $\vec{\vec{M}}$. We dub this the "Fourier Division Companion Matrix", or FDCM for short, because it is derived using trigonometric polynomial division. We show through examples that the algorithm computes both real and complex-valued roots, even double roots, to near machine precision accuracy.

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**Key words**: Fourier series, trigonometric polynomial, rootfinding, secular, companion matrix.

## 1. Introduction

More than a century ago, Frobenius showed that the roots of a polynomial could be found as the eigenvalues of a so-called "companion matrix" whose elements are trivial functions of the coefficients of the polynomial in the monomial basis. Similar companion matrices are now known to find the zeros of truncated series of Chebyshev, Legendre, Gegenbauer, Hermite and Bernstein polynomials as reviewed in [2].

A trigonometric polynomial of degree $N$ is a truncated Fourier series of the form $f_N(t) \equiv \sum_{j=0}^{N} a_j \cos(jt) + \sum_{j=1}^{N} b_j \sin(jt)$. It is known that such a trigonometric polynomial can always be converted by the change of coordinate $z = \exp(it)$ into an ordinary

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polynomial in $z$ with complex-valued coefficients. The zeros can then be computed by finding eigenvalues $z_k$ of the Frobenius companion matrix with complex-valued coefficients and applying $t_k = -i \log(z_k)$.

In this note, we show that it is possible to obtain a companion matrix for a truncated Fourier series directly. This provides a simple way to find the zeros of a function represented by its Fourier expansion. The determination of the maxima and minima and inflection points of a function are also problems in rootfinding because these points are the zeros of the first or second derivative of the function, and these derivatives can easily be found in Fourier form by term-by-term differentiation of the Fourier expansion for $f(t)$.

Trigonometric root finding problems arise in many applications. For example, computing the intersection of two curves is a common task in computer graphics. If one curve is specified implicitly as the zero set ("affine variety") of a bivariate algebraic polynomial $P(x, y)$ and the other is a closed curve, parameterized by a pair of trigonometric polynomials, the intersection problem may be reduced to finding zeros of a trigonometric polynomial. If the parameterized curve is specified by some functions $(x(t), y(t))$, then the univariate trigonometric polynomial whose roots are needed is $f(t) \equiv P(x(t), y(t))$. Later, we thus compute the intersection of an algebraic curve (a trifolium) with a parameterized ellipse.

## 2. Previous work on computing the zeros of trigonometric polynomials

Three transformations have been used to convert trigonometric polynomials into algebraic polynomials so that the standard rootfinders for the latter can be deployed. Weidner set $z = \exp(it)$, which yields a polynomial with complex coefficients and maps the real zeros in $t$ to roots on the unit circle in $z$ [18]. Schweikard avoided complex coefficients by the substitutions

$$t = 2\arctan(s) \qquad \longleftrightarrow \qquad s = \tan(t/2), \tag{2.1}$$

which convert a trigonometric polynomial to a rational function and then, after clearing denominators, to a polynomial in $s$ via the identities [16, 17]

$$\cos(t) = \frac{1-s^2}{1+s^2}, \qquad \sin(t) = \frac{2s}{1+s^2}. \tag{2.2}$$

Lastly, one may write $\cos(t) = c, \sin(t) = s$ and add the constraint $Q(c,s) \equiv c^2 + s^2 - 1 = 0$ which yields a system of two algebraic equations in the unknowns $(c,s)$. This option is very popular in robotics and yields the ECM companion matrix [4].

Other authors have applied interval arithmetic [7–9, 17] and the Durand-Kerner iteration for finding all roots simultaneously [1, 7–9, 13, 14]. Earlier and complementary companion matrix studies by the author include [2–4, 6].

We omit a full-scale review because we have already provided one in [2].