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A Hybrid Immersed Interface Method for Driven Stokes Flow in an Elastic Tube

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Abstract. We present a hybrid numerical method for simulating fluid flow through a compliant, closed tube, driven by an internal source and sink. Fluid is assumed to be highly viscous with its motion described by Stokes flow. Model geometry is assumed to be axisymmetric, and the governing equations are implemented in axisymmetric cylindrical coordinates, which capture 3D flow dynamics with only 2D computations. We solve the model equations using a hybrid approach: we decompose the pressure and velocity fields into parts due to the surface forcings and due to the source and sink, with each part handled separately by means of an appropriate method. Because the singularly-supported surface forcings yield an unsmooth solution, that part of the solution is computed using the immersed interface method. Jump conditions are derived for the axisymmetric cylindrical coordinates. The velocity due to the source and sink is calculated along the tubular surface using boundary integrals. Numerical results are presented that indicate second-order accuracy of the method.

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Key words: Stokes flow, interface tracking, immersed interface methods, axisymmetric cylindrical coordinates, boundary integrals.

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1. Introduction

Many advanced computational techniques have been developed for simulating the motion of an incompressible fluid interacting with flexible immersed structures, often with an eye toward biological applications. Much of the work has been inspired by Peskin's immersed boundary method [17], proposed originally for studying blood flow through a beating heart [16]. The immersed boundary method has since been applied in a wide variety of settings, e.g., [1, 3–6, 8, 9].

The immersed boundary method transfers singular forces from a boundary, or other structure, onto ambient fluid using smooth approximate Dirac delta functions, typically

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with $\mathscr{O}(h)$ support. Rather than capturing the jump discontinuity in the solution (e.g., pressure) at the immersed boundary, this approach approximates the solution as a continuous function with large gradient. In general, the immersed boundary method computes approximations with first-order spatial accuracy.

An alternative approach that captures the jumps in the solution and its derivatives sharply, and that generates approximations with second-order accuracy, is the immersed interface method developed by LeVeque and Li [11,12]. The immersed interface method is similar to a method developed earlier by Mayo for solving elliptic problems on an irregular domain [14,15]. Both methods are second-order Cartesian grid methods (though higher-order immersed interface methods have been developed [7,13]), with the key idea being the incorporation of known jumps in the solution or its derivatives into the finite difference schemes.

In this work we consider incompressible Stokes flow through a compliant, closed tube, driven by an internal source and sink. The internal source and sink allow us to represent inflow and outflow conditions while using techniques that apply to closed boundaries. Motivated by applications to blood flow through vessels, we take the structure and flow to be axisymmetric.

Our approach is to decompose the pressure and velocity fields into parts due to the tube boundary and due to the source and sink, so that each may be treated with an appropriate method. The tube surface creates a singularly-supported force on the fluid, resulting in an unsmooth solution; we find this part of the solution using the immersed interface method. Meanwhile, the smooth solution due to the source and sink is efficiently calculated along the tube surface via a boundary integral.

This paper is organized as follows: The axisymmetric governing equations for the fluid are detailed in Section 2 below. We describe our hybrid approach, incorporating the immersed interface method and boundary integral method, in Section 3. The jump conditions critical to the immersed interface method are derived for the axisymmetric setting in Section 4, and numerical results are presented in Section 5. Summary and directions for future work are presented in the Section 6.

2. Governing equations

Our aim is to simulate driven Stokes flow in a three-dimensional elastic tube. To take advantage of boundary integral solutions, we model the tube wall, Γ, as a closed surface, e.g., an ellipsoid or closed tube. To represent inflow and outflow conditions in this closed domain, we incorporate an internal source and sink, located at the ends of the tube (see Fig. 1). The impermeable tube is immersed in fluid in the computational domain Ω. In this study, the characteristics of the fluid (i.e., the viscosity) are assumed to be identical inside and outside of the tube.

In the immersed interface method, the fluid velocity and pressure are computed on a fixed, Eulerian grid, while a moving, Lagrangian frame of reference is used to track the location of the interface Γ over time. Our Eulerian grid is described by cylindrical, axisymmetric coordinates. That is, we use coordinates (r,θ,z), but assume that the domain