Numer. Math. Theor. Meth. Appl.
doi: 10.4208/nmtma.2013.1235nm

Vol. **6**, No. 4, pp. 617-636
November 2013

An h-Adaptive Runge-Kutta Discontinuous Galerkin Method for Hamilton-Jacobi Equations

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Received 2 November 2012; Accepted (in revised version) 26 February 2013

Available online 2 October 2013

Abstract. In [35,36], we presented an h-adaptive Runge-Kutta discontinuous Galerkin method using troubled-cell indicators for solving hyperbolic conservation laws. A tree data structure (binary tree in one dimension and quadtree in two dimensions) is used to aid storage and neighbor finding. Mesh adaptation is achieved by refining the troubled cells and coarsening the untroubled "children". Extensive numerical tests indicate that the proposed h-adaptive method is capable of saving the computational cost and enhancing the resolution near the discontinuities. In this paper, we apply this h-adaptive method to solve Hamilton-Jacobi equations, with an objective of enhancing the resolution near the discontinuities of the solution derivatives. One- and two-dimensional numerical examples are shown to illustrate the capability of the method.

AMS subject classifications: 65M60, 65M99, 35L65

Key words: Runge-Kutta discontinuous Galerkin method, h-adaptive method, Hamilton-Jacobi equation.

1. Introduction

In this paper, we present an h-adaptive Runge-Kutta discontinuous Galerkin (RKDG) method for solving Hamilton-Jacobi (H-J) equations

$$\begin{cases} \phi_t + H(\nabla_x \phi) = 0, \\ \phi(x,0) = \phi_0(x), \end{cases} \tag{1.1}$$

where $x = (x_1, \cdots, x_d) \in \mathbf{R}^d$, $t > 0$. H-J equations are of practical importance with applications ranging from optimal control and differential games to geometric optics and image processing. Viscosity solutions of H-J equations are studied [11,12] to single out a unique,

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practically relevant solution. These viscosity solutions are Lipschitz continuous, and may develop discontinuous derivatives no matter how smooth the initial conditions are.

The study of numerical approximations to the viscosity solution of H-J equation (1.1) was started by Crandall and Lions. In [12] they proposed a monotone finite difference scheme and proved convergence to the viscosity solution. Unfortunately, monotone schemes can be at most first order accurate. Osher and Sethian [22] and Osher and Shu [23] constructed a class of high order ENO (essentially non-oscillatory) schemes which were adapted from ENO schemes in [13, 30, 31] for hyperbolic conservation laws. Their construction was based on the observation that there is a close relation between H-J equations and conservation laws, and as a result successful numerical methods for conservation laws can be adapted for solving H-J equations. For example, WENO (weighted ENO) schemes in [16, 21], Hermite WENO schemes in [24, 25] and RKDG method in [8, 9] were adapted for solving H-J equations by Jiang and Peng [15], Qiu and Shu [26] and Hu and Shu [14] (see also the reinterpretation work [19]), respectively. For other approaches, we mention the work of central high resolution schemes developed in [2, 3, 18, 20] and two different direct DG (discontinuous Galerkin) methods introduced respectively in [5] and [33].

In the traditional way numerical methods adopt fixed and pre-assigned meshes, so one has to develop higher-order (third, fourth or ever higher) numerical schemes in order to enhance the resolution of the numerical approximations. Lower-order schemes, which may be rather simple, can also produce high resolution with small number of mesh points if mesh adaptation is employed. So far a few works have been done on adaptive algorithms for H-J equation (1.1). We refer, for instance, to Tang et al. [32], where an adaptive mesh redistribution (r-adaptive) method was developed for solving two- and three-dimensional H-J equations. We also refer to Cockburn and his collaborators [1, 4].

The singularities in the solution derivatives cause great difficulties in obtaining numerical solutions of H-J equations. Local error at the discontinuities of the derivatives may be significantly larger and dominate the global error. Higher-order elements at discontinuities can not decrease the local error but may result in oscillatory solutions, so local mesh refinement is a good solution which can enhance the resolution by steepening the discontinuities. This motivates us to work on the h-adaptive method for H-J equations.

Our h-adaptive method for H-J equations proposed in this paper is based on the RKDG methods. The RKDG methods for solving hyperbolic conservation laws are high-order accurate and highly parallelizable methods which can easily handle complicated geometries, boundary conditions and $h-p$ adaptivity. These methods have made their way into the main stream of computational fluid dynamics and other areas of applications. The first DG method was introduced in 1973 by Reed and Hill [28] for the neutron transport problem. A major development of this method was carried out by Cockburn et al. in a series of papers [6–9], in which a framework to solve nonlinear time dependent hyperbolic conservation laws was established. They adopted explicit, nonlinearly stable high order Runge-Kutta time discretizations [30], DG space discretizations with exact or approximate Riemann solvers as interface fluxes and TVB (total variation bounded) nonlinear limiter [29] to achieve nonoscillatory properties, and the method was termed as RKDG method. Detailed description of the method as well as its implementation can be found in the review