Numer. Math. Theor. Meth. Appl.
doi: 10.4208/nmtma.2013.1230nm

Vol. **6**, No. 4, pp. 637-656
November 2013

# Error Estimates and Superconvergence of RT0 Mixed Methods for a Class of Semilinear Elliptic Optimal Control Problems

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Received 24 September 2012; Accepted (in revised version) 19 December 2012

Available online 2 October 2013

**Abstract.** In this paper, we will investigate the error estimates and the superconvergence property of mixed finite element methods for a semilinear elliptic control problem with an integral constraint on control. The state and co-state are approximated by the lowest order Raviart-Thomas mixed finite element and the control variable is approximated by piecewise constant functions. We derive some superconvergence properties for the control variable and the state variables. Moreover, we derive $L^{\infty}$- and $H^{-1}$-error estimates both for the control variable and the state variables. Finally, a numerical example is given to demonstrate the theoretical results.

**AMS subject classifications**: 49J20, 65N30

**Key words**: Semilinear elliptic equations, optimal control problems, superconvergence, error estimates, mixed finite element methods.

## 1. Introduction

As far as we know, there have been extensive studies in superconvergence of finite element approximations for optimal control problems, see, for example, [6, 17–19, 21, 24] for standard finite element methods and [4, 5, 8, 9, 20] for mixed finite element methods. In [21], Meyer and Rösch constructed a postprocessing projection operator and derived a quadratic superconvergence of the control by finite element methods. In [18], Liu and Yan considered recovery type superconvergence and a posteriori error estimates for control problem governed by Stokes equations. Next, Yan [24] analyzed the superconvergence property of finite element method for an optimal control problem governed by integral equations. A priori error estimates and superconvergence for an optimal control problem

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of bilinear type are obtained in [23]. Compared with standard finite element methods, the mixed finite element methods have many advantages. When the objective functional contains gradient of the state variable, we will firstly choose the mixed finite element methods. In [5], we used the postprocessing projection operator, which was defined by Meyer and Rösch (see [21]) to prove a quadratic superconvergence of the control by mixed finite element methods. We derived error estimates and superconvergence of mixed methods for convex optimal control problems in [9]. But in that paper, the convergence order is $h^{3/2}$ since the analysis was restricted by the low regularity of the control. Recently, in [8] we derived superconvergence and $L^\infty$-error estimates of RT1 mixed finite element methods for semilinear elliptic control problems with an integral control constraint, however, we didn't considered the superconvergence property of the vector functions.

The goal of this paper is to derive the superconvergence property, the $L^\infty$-error estimates and the $H^{-1}$-error estimates of the lowest order mixed finite element approximation for a semilinear elliptic control problem with an integral control constraint. Firstly, we derive the superconvergence property between average $L^2$ projection and the approximation of the control variable, the convergence order is $h^2$ instead of $h^{3/2}$ in [9], which is caused by the different admissible set. Then, we will derive some superconvergence properties for the state variables. We also derive the $L^\infty$-error estimates for both the control variable and the state variables. Next, we give some applications of the superconvergence results. We derive a superconvergence result for the control variable by using a recovery operator instead of a projection of the discrete adjoint state $z_h$ in the reference [8]. Furthermore, we shall obtain $H^{-1}$-error estimates for both the control variable and the state variables. Finally, we present a numerical experiment to demonstrate the practical side of the theoretical results about superconvergence and $L^\infty$-error estimates.

We consider the following semilinear optimal control problems for the state variables $\boldsymbol{p}$, $y$, and the control $u$ with an integral control constraint:

$$\min_{u\in U_{ad}} \left\{ \frac{1}{2}\|\boldsymbol{p}-\boldsymbol{p}_d\|^2 + \frac{1}{2}\|y-y_d\|^2 + \frac{\nu}{2}\|u\|^2 \right\} \tag{1.1}$$

subject to the state equation

$$-\mathrm{div}(A(x)\mathbf{grad}y) + \phi(y) = Bu, \quad x\in\Omega, \tag{1.2}$$

which can be written in the form of the first order system

$$\mathrm{div}\boldsymbol{p} + \phi(y) = Bu, \quad \boldsymbol{p} = -A(x)\mathbf{grad}y, \quad x\in\Omega, \tag{1.3}$$

and the boundary condition

$$y = 0, \quad x\in\partial\Omega, \tag{1.4}$$

where $\Omega$ is a rectangular domain in $\mathbb{R}^2$. $U_{ad}$ denotes the admissible set of the control variable, defined by

$$U_{ad} = \left\{ u\in L^\infty(\Omega): \int_\Omega u dx \geq 0 \right\}. \tag{1.5}$$