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Alternating Direction Implicit Galerkin Finite Element Method for the Two-Dimensional Time Fractional Evolution Equation

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Abstract. New numerical techniques are presented for the solution of the two-dimensional time fractional evolution equation in the unit square. In these methods, Galerkin finite element is used for the spatial discretization, and, for the time stepping, new alternating direction implicit (ADI) method based on the backward Euler method combined with the first order convolution quadrature approximating the integral term are considered. The ADI Galerkin finite element method is proved to be convergent in time and in the L^2 norm in space. The convergence order is $\mathscr{O}(k|\ln k|+h^r)$, where k is the temporal grid size and h is spatial grid size in the x and y directions, respectively. Numerical results are presented to support our theoretical analysis.

AMS subject classifications: 65M06, 65M12, 65M15, 65M60

Key words: Fractional evolution equation, alternating direction implicit method, Galerkin finite element method, backward Euler.

1. Introduction

We study an alternating direction implicit (ADI) method for the numerical solution of the fractional evolution equation

$$u_t-\int_0^t \beta(t-s)\Delta u(x,y,s)ds=f(x,y,t), \quad (x,y)\in\Omega, \quad t\in J, \tag{1.1}$$

with boundary and initial conditions

$$u(x,y,t)=0, \qquad (x,y)\in\partial\Omega, \quad t\in J, \tag{1.2a}$$

$$u(x,y,0)=u^0(x,y), \qquad (x,y)\in\Omega\times\partial\Omega, \tag{1.2b}$$

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respectively, where $u_t = \partial u/\partial t$, Δ is the two-dimensional Laplacian, $J=(0,T]$, $\Omega = R\times R$, $R=(0,1)$, $\partial\Omega$ is the boundary of the domain Ω, the kernel $\beta(t)$ is assumed to be $t^{-1/2}$, $u^0(x,y)$ and $f(x,y,t)$ are given functions.

This equation possesses the remarkable feature that it may be considered as an equation intermediate between the standard (parabolic) heat equation and the (hyperbolic) wave equation. In fact, the integral operator $I^{1/2}$ that maps each (locally integrable) function $\varphi(t)$, $t>0$, into the function

$$(I^{1/2}\varphi)(t) = \int_0^t (t-s)^{-1/2}\varphi(s)ds$$

is such that

$$(I^{1/2}(I^{1/2}\varphi))(t) = \pi\int_0^t \varphi(s)ds =: \pi(I\varphi)(t).$$

In recent decade years, more and more attentions have been placed on the development and research of fractional differential and integral equations, because they can describe many phenomenons, physical and chemical process more accurately than classical integer-order differential equations and have been widely used in many other fields, such as viscoelasticity [1,2], biological systems, finance [3], hydrology [4] and so on. Therefore, it is important to find some efficient methods to solve fractional differential and integral equations. Many considerable works on the theoretical analysis [5–8] have been carried on, but analytic solutions of most fractional differential equations cannot be obtained explicitly. Many authors have resorted to numerical solution strategies based on convergence and stability analysis [9–34]. For one-dimensional problems, Lopez-Marcos [10] and Tang [12] analyzed finite difference schemes for a partial integro-differential equation. Lin and Xu [16] considered numerical approximations based on a finite difference scheme in time and Legendre spectral methods in space. Li and Xu [17] extended their previous work and proposed a spectral method in both temporal and spatial discretizations. Deng [18] developed the finite element method for the numerical solution of the space-time fractional Fokker-Planck equation. Li and Xu [24] constructed and analyzed stable and high order scheme to solve the integro-differential equation which is discretized by the finite difference in time and by the finite element method in space.

For two-dimensional problems, Zhang et al. [31] presented the finite difference/element method for a two-dimensional modified fractional diffusion equation. They used the second-order backward differentiation formula in time and the finite element method in space. Ji and Tang [32] considered the Runge-Kutta and discontinuous Galerkin (DG) methods for the fractional diffusion equations. Chen and Liu [34] proposed a second-order accurate numerical method for the two-dimensional fractional advection-dispersion equation. This method combined the alternating directions implicit approach with the unshifted Grünwald formula for the advection term, the right-shifted Grünwald formula for the diffusion term, and a Richardson extrapolation to establish an unconditionally stable second order accurate difference scheme.

The numerical solutions of the high-dimensional fractional evolution equations are still a challenge. The purpose in this paper is to consider effective numerical methods