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Domain Decomposition Preconditioners for Discontinuous Galerkin Discretizations of Compressible Fluid Flows

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Abstract. In this article we consider the application of Schwarz-type domain decomposition preconditioners to the discontinuous Galerkin finite element approximation of the compressible Navier-Stokes equations. To discretize this system of conservation laws, we exploit the (adjoint consistent) symmetric version of the interior penalty discontinuous Galerkin finite element method. To define the necessary coarse-level solver required for the definition of the proposed preconditioner, we exploit ideas from composite finite element methods, which allow for the definition of finite element schemes on general meshes consisting of polygonal (agglomerated) elements. The practical performance of the proposed preconditioner is demonstrated for a series of viscous test cases in both two- and three-dimensions.

AMS subject classifications: 65F08, 65N12, 65N15, 65N30

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1. Introduction

The application and development of discontinuous Galerkin finite element methods (DGFEMs) for the numerical approximation of the compressible Euler and Navier-Stokes equations has been considered extensively within the current literature; for example, see [12–15, 18, 21, 22, 25, 27–29, 33–35], and the references cited therein. DGFEMs have several important advantages over well established finite volume methods. The concept of higher-order discretization is inherent to the DGFEM. The stencil

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is minimal in the sense that each element communicates only with its direct neighbours. In particular, in contrast to the increasing stencil size needed to increase the accuracy of classical finite volume methods, the stencil of DGFEMs is the same for any order of accuracy, which has important advantages for the implementation of boundary conditions and for the parallel efficiency of the method. Moreover, due to the simple communication at element interfaces, elements with so-called hanging nodes can be easily treated, a fact that simplifies local mesh refinement (h-refinement). Additionally, the communication at element interfaces is identical for any order of the method, which simplifies the use of methods with different polynomial orders p in adjacent elements. This allows for the variation of the order of polynomials over the computational domain (p-refinement), which in combination with h-refinement leads to hp-adaptivity. It should also be noted that the use of standard conforming methods for the discretization of compressible flows suffer from issues concerning numerical stability, which must be tackled with the introduction of suitable numerical dissipation terms in the form of SUPG stabilisation or artificial viscosity. Such terms can adversely affect the convergence of the underlying iterative solver used to compute the numerical solution.

Despite the advantages and capabilities of the DGFEM, the method is not yet mature and current implementations are subject to strong limitations for its application to large scale industrial problems. This situation is clearly reflected by the breadth of research activity and the increasing number of scientific articles concerning DGFEMs. In particular, one of the key issues is the design of efficient strategies for the solution of the system of equations generated by a DGFEM, which we should point out is typically larger than the corresponding matrix system generated when a conforming finite element method is employed. However, in the context of p-version finite element methods, it has been shown in the recent article [17] that DGFEMs can indeed outperform their conforming counterparts in the sense that the former class of methods may be more accurate for a given number of degrees of freedom as the polynomial degree is increased. For two-dimensional problems, parallel direct solvers such as MUMPS [1–3], for example, are generally applicable. However, for such problems, they still require very large amounts of memory in order to store the L and U factors. Moreover, for three-dimensional calculations, direct methods become impractical. Thereby, in this setting iterative solvers, such as GMRES, for example, must be exploited. Of course, the key to computing the solution in an efficient manner relies on the choice of the underlying preconditioning strategy employed. In order to exploit the parallel capabilities of modern high performance computing architectures, it is natural to consider multilevel techniques, which are based on exploiting some form of domain decomposition approach, such as additive and multiplicative Schwarz preconditioners, since they are naturally highly-parallelizable and scalable to a large number of processors.

In the context of DGFEMs, recent work on the design and analysis of multilevel preconditioners for DGFEMs has been undertaken; for example, we refer to [4, 5, 8, 10, 16, 20]. In particular, in [4, 5], cf., also, [8], it was demonstrated that Schwarz-type preconditioners are particularly suited to DGFEMs, in the sense that uniform scalability of the underlying iterative method may be established without the need to overlap the