Numer. Math. Theor. Meth. Appl. Vol. 7, No. 2, pp. 179-192
doi: 10.4208/nmtma.2014.y12038 May 2014

Simple Fourth-Degree Cubature Formulae with Few Nodes over General Product Regions

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Received 15 October 2012; Accepted (in revised version) 23 September 2013

Available online 16 May 2014

Abstract. A simple method is proposed for constructing fourth-degree cubature formulae over general product regions with no symmetric assumptions. The cubature formulae that are constructed contain at most $n^2 + 7n + 3$ nodes and they are likely the first kind of fourth-degree cubature formulae with roughly n^2 nodes for non-symmetric integrations. Moreover, two special cases are given to reduce the number of nodes further. A theoretical upper bound for minimal number of cubature nodes is also obtained.

AMS subject classifications: 65D30, 65D32

Key words: Fourth-degree cubature formula, cubature formula, product region, non-symmetric region, numerical integration.

1. Introduction

We are interested in the integration

$$I(f) = \int_\Omega f(\boldsymbol{x})\boldsymbol{\rho}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \tag{1.1}$$

over the product region

$$\Omega = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \tag{1.2}$$

with the non-negative weight function $\boldsymbol{\rho}(\boldsymbol{x})$ in the product form

$$\boldsymbol{\rho}(\boldsymbol{x}) = \rho_1(x_1) \cdots \rho_n(x_n), \tag{1.3}$$

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where a_i and b_i are finite or infinite numbers. For a general smooth function $f(\boldsymbol{x})$, such an integration is often numerically approximated by the following weighted sum

$$Q(f) = \sum_{j=1}^{N} w^{(j)} f(\boldsymbol{x}^{(j)}), \tag{1.4}$$

where $\boldsymbol{x}^{(j)} = \left(x_1^{(j)}, x_2^{(j)}, \cdots, x_n^{(j)}\right) \in \Omega \subset \mathbb{R}^n$ are N distinct cubature nodes for $j = 1, 2, \cdots, N$, and $w^{(j)} \in \mathbb{R}$ are cubature weights. Denote by $\mathcal{P}_m^n$ the space of the polynomials in n variables of degree no more than m. $Q(f)$ in (1.4) is said to be of degree m with respect to $I(f)$, if $Q(f) = I(f)$ for any $f \in \mathcal{P}_m^n$ and $Q(g) \neq I(g)$ for at least one $g \in \mathcal{P}_{m+1}^n$.

From the numerical points of view, people are interested in the cubature formula with a minimal number of nodes. Denote by $N_{min}^G(m,n)$ the minimal number of nodes of cubature formulae of degree m over general n-dimensional regions. Then one has the following general lower bound (see [3, Th. 9])

$$N_{\min}^G(m,n) \geq \dim \mathcal{P}_{[m/2]}^n, \tag{1.5}$$

where $[x]$ denotes the integer part of x. This lower bound is not very sharp for $n \geq 2$ and can be improved for odd degrees as follows:

$$N_{\min}^G(2k+1,n) \geq \dim \mathcal{P}_k^n + \frac{\sigma_l}{l}, \tag{1.6}$$

where, uniformly for any integer l satisfying $2 \leq l \leq n$, the constant

$$\begin{aligned} \sigma_l := \dim &\left\{ (f_1(\boldsymbol{x}), \cdots, f_l(\boldsymbol{x})) \in \mathcal{Z}_{k+1}^l : \ \sum_{i=1}^{l} x_i f_i(\boldsymbol{x}) \in \mathcal{P}_{k+1}^n \right\} \\ &- \dim \left\{ (f_1(\boldsymbol{x}), \cdots, f_l(\boldsymbol{x})) \in \mathcal{Z}_{k+1}^l : \ \sum_{i=1}^{l} x_i f_i(\boldsymbol{x}) \in \mathcal{Z}_{k+1} \right\}, \end{aligned}$$

and

$$\mathcal{Z}_{k+1} := \{ f(\boldsymbol{x}) \in \mathcal{P}_{k+1}^n : \ g(\boldsymbol{x}) \in \mathcal{P}_k^n \Rightarrow I(fg) = 0 \}.$$

See [2, 6, 7]. For centrally symmetric regions, one can get a better lower bound

$$N_{\min}^{CS}(2k+1,n) \geq 2 \dim \mathcal{Q}_k^n - \begin{cases} 1, & \text{if } k \text{ is even and } \mathbf{0} \text{ is a node}, \\ 0, & \text{others}, \end{cases} \tag{1.7}$$

where $\mathcal{Q}_{2k}^n$ is the subspace of $\mathcal{P}_{2k+1}^n$ generated by even polynomials and $\mathcal{Q}_{2k+1}^n$ is the subspace of $\mathcal{P}_{2k+1}^n$ generated by odd polynomials (see [7]), or explicitly (see [5])

$$N_{\min}^{CS}(2k+1,n) \geq \begin{cases} \binom{n+k}{n} + \sum_{s=1}^{n-1} 2^{s-n} \binom{s+k}{s}, & \text{if } k \text{ is odd}, \\ \binom{n+k}{n} + \sum_{s=1}^{n-1} (1-2^{s-n}) \binom{s+k-1}{s}, & \text{if } k \text{ is even}. \end{cases} \tag{1.8}$$