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Accuracy Enhancement of Discontinuous Galerkin Method for Hyperbolic Systems

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Abstract. We study the enhancement of accuracy, by means of the convolution post-processing technique, for discontinuous Galerkin(DG) approximations to hyperbolic problems. Previous investigations have focused on the superconvergence obtained by this technique for elliptic, time-dependent hyperbolic and convection-diffusion problems. In this paper, we demonstrate that it is possible to extend this post-processing technique to the hyperbolic problems written as the Friedrichs' systems by using an upwind-like DG method. We prove that the L_2-error of the DG solution is of order $k+1/2$, and further the post-processed DG solution is of order $2k+1$ if Q_k-polynomials are used. The key element of our analysis is to derive the $(2k+1)$-order negative norm error estimate. Numerical experiments are provided to illustrate the theoretical analysis.

AMS subject classifications: 65N30, 65M60

Key words: Discontinuous Galerkin method, hyperbolic problem, accuracy enhancement, post-processing, negative norm error estimate.

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1. Introduction

In this paper, we consider an upwind-like DG method for solving the hyperbolic problem written as the Friedrichs' systems [7],

$$\sum_{i=1}^{d} A_i \partial_i \mathbf{u} + B\mathbf{u} = \mathbf{f}, \quad \text{in} \ \ \Omega, \quad (M - D_n)\mathbf{u} = \mathbf{0}, \quad \text{on} \ \ \partial\Omega, \tag{1.1}$$

where Ω is a bounded domain in R^d, matrix $D_n = \sum_{i=1}^{d} A_i n_i$, $n = (n_1, \cdots, n_d)^T$ is the outward unit normal vector. Our main aim is to show that it is possible to enhance the accuracy of this DG approximation by using the convolution post-processing technique. This post-processing technique was originally introduced by Bramble and Schatz [1]

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using continuous finite element methods for elliptic equations and later was developed by Cockburn, Luskin, Shu and S$\ddot{u}$li [2] using the DG method for time-dependent hyperbolic equations, and further by Ji, Xu and Ryan [9] using the LDG method for time-dependent convection-diffusion equations. The post-processing technique is carried out by a convolution operation applied to the finite element solution. The key of this technique is to derive a superconvergence order (for example, the $O(h^{2k+1})$-order) error estimate in the negative norm for the finite element solution. By the post-processing, the order of error in the L_2-norm can be enhanced up to the order of error in the negative norm. Some other post-processing techniques [10,11,20,21] also have been proposed in enhancing the accuracy of the finite element solutions, but they do not possess such high accuracy.

DG methods for solving problem (1.1) basically can be classified as both the numerical flux method and the penalty method, see [3,5,6,13,19,22] and the references therein. In the numerical flux method, the key element is to choose the numerical trace $D_n\widehat{\mathbf{u}}$ properly in the weak form of problem (1.1):

$$-\int_K \mathbf{u}\cdot\sum_{i=1}^d A_i\partial_i\mathbf{v}dx+\int_K(B-\sum_{i=1}^d\partial_iA_i)\mathbf{u}\cdot\mathbf{v}dx+\int_{\partial K}D_n\widehat{\mathbf{u}}\cdot\mathbf{v}ds=\int_K\mathbf{f}\cdot\mathbf{v}dx, \quad (1.2)$$

where K is the element. In the traditional upwind scheme (see [8,13,19]), the numerical trace is defined by first splitting matrix $D_n=\sum A_in_i$ into the symmetric form $D_n=A^++A^-$ with $A^+\geq 0$ (positive semi-definite) and $A^-\leq 0$ (negative semi-definite), and then setting the numerical trace $D_n\widehat{\mathbf{u}}|_{\partial K}=A^+\mathbf{u}^++A^-\mathbf{u}^-$, where $\mathbf{u}^+$ and $\mathbf{u}^-$ are the traces of $\mathbf{u}$ on ∂K from the interior and exterior of K, respectively. In this paper, we will present an upwind-like DG scheme which is slightly different from the traditional one. We first decompose each A_i into $A_i=A_i^++A_i^-$, and then define the numerical trace by setting $D_n\widehat{\mathbf{u}}=\sum_{i=1}^dA_i^+n_i\widehat{\mathbf{u}}+\sum_{i=1}^dA_i^-n_i\widehat{\mathbf{u}}$, and $A_i^\pm n_i\widehat{\mathbf{u}}=A_i^\pm n_i\mathbf{u}^+(A_i^\pm n_i\mathbf{u}^-)$ if $A_i^\pm n_i\geq 0(A_i^\pm n_i\leq 0)$. The advantage of our method is that the splitting can be implemented only once before the triangulation is made, while in the traditional method, since matrix D_n depends on the boundary normal vector $n|_{\partial K}$, then for each element K and each face $\mathcal{F}_K\subset\partial K$, we always need to split $D_n|_{\mathcal{F}_K}=A^++A^-$. Therefor, such splitting is very consuming in practical computation. More importantly, for this DG method, we can derive the $(2k+1)$-order error estimates in the negative norm. It should be point out that Cockburn et al. in [2] (also see [9,14]) have established a framework to prove the negative norm error estimates for DG methods applied to time-dependent hyperbolic problems, but their analysis is very relied on the time-dependent structure of the problem and is not available to time-independent hyperbolic problem (1.1). In this paper, by means of the a priori error estimate in a mesh-dependent norm and the dual argument technique, we derive the desired negative norm error estimate which allows us to enhance the accuracy of DG solution from $(k+1/2)$-order to $(2k+1)$-order in the L_2-norm by using the convolution post-processing technique.

Throughout this paper, let Ω be a bounded open polyhedral domain in R^d, $d\geq 2$. For any open subset $\mathcal{D}\subset\Omega$ and integers $m\geq 0$, we denote by $H^m(\mathcal{D})$ the usual Sobolev