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Evaluating Local Approximations of the L^2-Orthogonal Projection Between Non-Nested Finite Element Spaces

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Abstract. We present quantitative studies of transfer operators between finite element spaces associated with unrelated meshes. Several local approximations of the global L^2-orthogonal projection are reviewed and evaluated computationally. The numerical studies in 3D provide the first estimates of the quantitative differences between a range of transfer operators between non-nested finite element spaces. We consider the standard finite element interpolation, Clément's quasi-interpolation with different local polynomial degrees, the global L^2-orthogonal projection, a local L^2-quasi-projection via a discrete inner product, and a pseudo-L^2-projection defined by a Petrov-Galerkin variational equation with a discontinuous test space. Understanding their qualitative and quantitative behaviors in this computational way is interesting per se; it could also be relevant in the context of discretization and solution techniques which make use of different non-nested meshes. It turns out that the pseudo-L^2-projection approximates the actual L^2-orthogonal projection best. The obtained results seem to be largely independent of the underlying computational domain; this is demonstrated by four examples (ball, cylinder, half torus and Stanford Bunny).

AMS subject classifications: 65D05, 65F10, 65N30, 65N50, 65N55

Key words: Finite elements, unstructured meshes, non-nested spaces, transfer operators, interpolation, projection.

1. Introduction

The question of how to interpolate functions in finite element spaces is as old as the finite element method itself. Approximation operators which map a given function to a finite element space appear frequently in numerical analysis for a variety of reasons. For both a priori and a posteriori discretization error estimates, one often needs to

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treat general elements of infinite dimensional function spaces. However, such operators are usually not evaluated computationally but merely used in the analysis. A famous example is the early work by Clément [26] on finite element quasi-interpolation of discontinuous (non-smooth) functions.

In this paper, we study different operators mapping to finite element spaces. Although a couple of the reviewed properties hold true in case the input function is in an infinite dimensional function space, our main focus is the information transfer between two finite element spaces. We investigate this by a series of numerical studies. The respective spaces are Lagrange conforming finite elements of first order associated with (either two or a whole hierarchy of) non-nested meshes. This non-nested information transfer is important, for instance, in non-conforming domain decomposition methods [8, 9, 41, 62, 63] or in domain decomposition or multigrid methods with non-nested coarse spaces [11, 15, 18, 20, 24, 30, 47, 48, 65] for the solution of partial differential equations. It appears both in the analysis and in practical computations.

We present computational results on the behavior of local approximations of the L^2-projection between non-nested finite element spaces. Our numerical studies are based on a detailed overview of different transfer operators. Let us emphasize that there is neither a conclusive characterization of the information transfer between finite element spaces associated with non-nested meshes nor a comprehensive classification of transfer operators in this setting yet. In the present investigations, we show that, apart from basic similarities, there are substantial conceptual and qualitative differences as well as substantial quantitative differences between the studied operators.

Our research is in part motivated by the fundamental work on quasi-interpolation [26, 54]. We also learned about advanced techniques for the construction of transfer operators from [41, 62, 63] in the context of non-conforming domain decomposition methods. Other interesting studies giving basic insights into the analysis of approximation operators in finite element spaces, which influenced our work, can be found in [4, 12, 13, 20, 24, 56–58, 64].

The evaluation criteria that are discussed in the theoretical part are the H^1-stability, an L^2-approximation property, the locality of the information transfer, and the projection properties. In the computational part, we study the mutual relations between the diverse transfer operators by numerical experiments. The operators are evaluated for four examples of computational domains (ball, cylinder, half torus and Stanford Bunny), each time for a series of independently generated meshes. All distances with respect to certain operator norms are computed by solving the corresponding generalized eigenvalue problems. To our knowledge, similar studies estimating distances between transfer operators in the present context cannot be found elsewhere.

Let us briefly comment on an application for the non-nested information transfer studied in this paper to multilevel preconditioners, which are among the most efficient algorithms for the solution of discretized partial differential equations in many applications; see, e.g., [5, 14, 17, 39, 40, 46, 50, 59, 66] for some of the most influential achievements. For applications in computational engineering involving complicated geometries in three dimensions, the construction of coarse spaces is often demanding