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Use of Shifted Laplacian Operators for Solving Indefinite Helmholtz Equations

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Abstract. A shifted Laplacian operator is obtained from the Helmholtz operator by adding a complex damping. It serves as a basic tool in the most successful multigrid approach for solving highly indefinite Helmholtz equations — a Shifted Laplacian preconditioner for Krylov-type methods. Such preconditioning significantly accelerates Krylov iterations, much more so than the multigrid based on original Helmholtz equations. In this paper, we compare approximation and relaxation properties of the Helmholtz operator with and without the complex shift, and, based on our observations, propose a new hybrid approach that combines the two. Our analytical conclusions are supported by two-dimensional numerical results.

AMS subject classifications: 65F10, 65N22, 65N55

Key words: Indefinite Helmholtz operator, multigrid, shifted Laplacian, ray correction.

1. Introduction

Considered here is a two-dimensional Helmholtz equation

$$Lu = \Delta u(x) + k^2(x)u(x) = f(x), \quad x \in \Omega \subset \mathbf{R}^2, \tag{1.1}$$

accompanied by the first-order Sommerfeld boundary conditions

$$\frac{\partial u(x)}{\partial n} - iku(x) = 0, \quad x \in \partial\Omega. \tag{1.2}$$

Use of standard discretization methods, considered on a sufficiently fine scale h, yields a system of linear equations

$$L^h u^h = f^h, \tag{1.3}$$

where $L^h \in \mathbf{C}^{n\times n}$ is a sparse matrix, with n, the number of discrete degrees of freedom, typically large.

Iterative methodologies applied to (1.3) include multigrid (MG) approach which is the focus of this paper. Multigrid is well known for its ability to deliver accurate solutions at optimal computational costs. The indefinite Helmholtz (HLM) operators, however, present many challenges for MG development, some of which are discussed later on and many are presented, for instance, in [1,9]. Multigrid approaches for (1.1) notably include [2, 3, 5, 10, 11, 13, 14, 17]. The most practical multigrid method to date is the Shifted Laplacian (SL) approach suggested in [5, 6] and further developed in [4, 7, 8, 15, 16] among others. The SL employs differential operators $M = L + ik^2\beta I$ which are obtained from the Helmholtz operator by adding a complex shift. The positive constant β is typically chosen to be one half. Discretization on scale h yields

$$M^h = L^h + ik^2\beta I^h. \tag{1.4}$$

Both discrete operators L^H and M^H, $H = h, 2h, \cdots$ are computed using second-order central differences, resulting in five-point stencils.

The M^h-based multigrid preconditioners provide a significant acceleration of Krylov iterations applied for solving (1.3). This is due to the fact that all eigenvalues of the preconditioned operator, $(M^h)^{-1}L^h$ are located at the right half of the complex plane, i..e, for each such eigenvalue, its real part is positive. The goal of this paper is to investigate the advantages and shortcomings of employing the shifted Laplacian and the Helmholtz operators on different scales, $H = h, 2h, \cdots$, and to see if their combined use results in an improved preconditioning.

We compare two sets of operators $\{L_H\}_{H=h,2h,\cdots}$ and $\{M_H\}_{H=h,2h,\cdots}$ from two perspectives:

- Approximate qualities of coarse operators;
- The *smoothing* and *convergence* rates of relaxation schemes, especially for the near-kernel and physically smooth error components, the latter being comprised of eigenvectors of L^h with negative eigenvalues.

The remainder of the paper is organized as follows. The near-kernel components (nkc) of (1.1) are presented in Section 2, and the Shifted Laplacian preconditioning on different scales is discussed in Section 3. The accuracy, Section 4, and the relaxation, Section 5, properties of the Helmholtz and the Shifted Laplacian operators are compared, and a multigrid preconditioner based on a combination of the two is proposed, Section 6. Numerical experiments are presented in Section 7, and the concluding remarks are given in Section 8.

2. Error components and the Helmholtz operator

An efficient multigrid algorithm works in the following way: Each coarse operator A^H, $H = 2h, 4h, \cdots$ approximates the finest operator A^h for all components unreduced by the processing on preceding finer grids. A finer error $e^{H/2}$ with a large relative